Functional renormalisation group for quantum many-body systems

Lecture 1:

The renormalisation group and functional methods for quantum many-body systems

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Objective and outline of these lectures

- In these lectures, we will study the **functional renormalisation group (FRG)** approach for **quantum many-body systems**.
- By the end of these lectures, we intend to develop a general understanding of the FRG approach, so that you can use it to study **current problems** in **many-body physics**.
- Particularly, we will see that the FRG is a **robust approach** for studying **strongly-correlated systems** and **critical phenomena**.

Objective and outline of these lectures

- **Lecture 1:**

  The renormalisation group and **functional methods** for **quantum many-body systems**.

- **Lecture 2:**

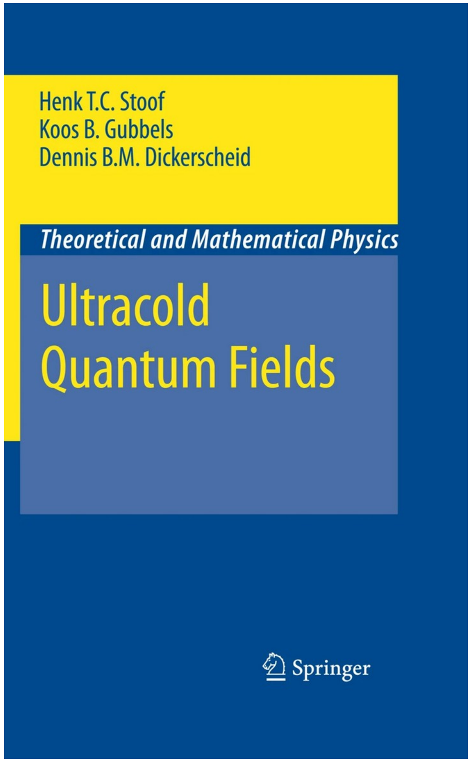
  Functional renormalisation group for the **O(2)-model**.

- **Lecture 3:**

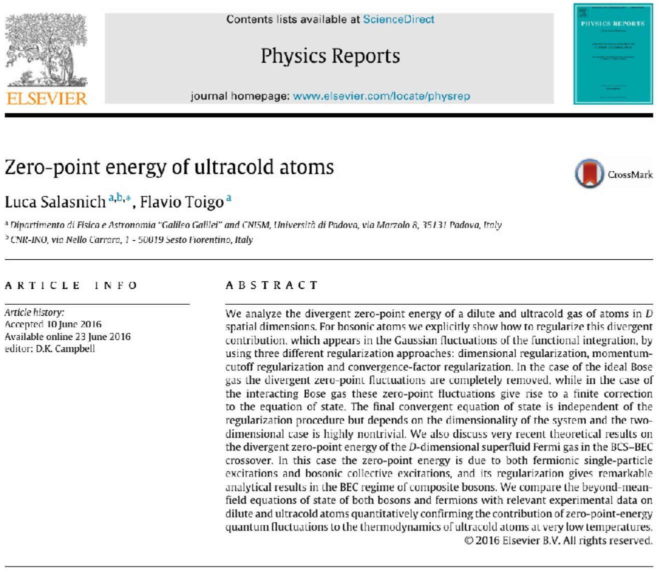
- **Functional renormalisation group** for **Bose** and **Fermi** gases.

Bibliography

- **Quantum field theory** and **functional methods** for many-body systems:



H. Stoof, K. B. Gubbels, and D. Dickerscheid, *Ultracold quantum fields* (Springer, 2009).



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Zero-point energy of ultracold atoms

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ABSTRACT

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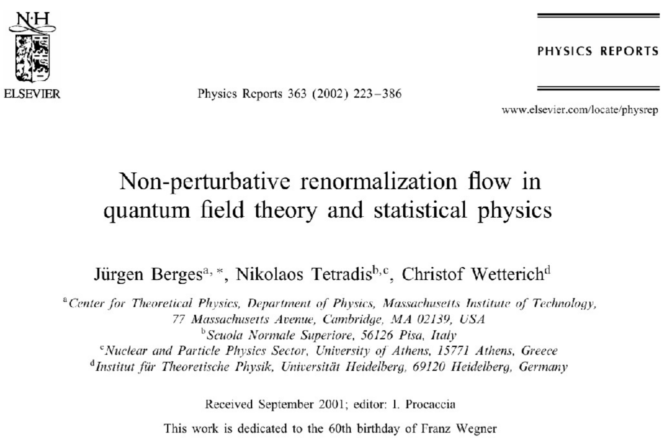
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Non-perturbative renormalization flow in quantum field theory and statistical physics

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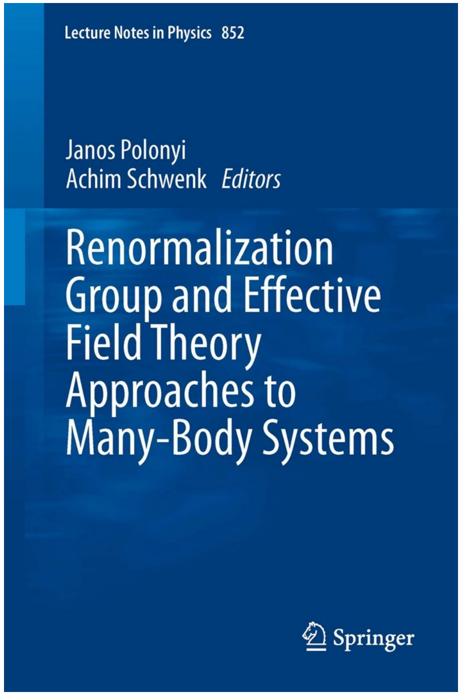
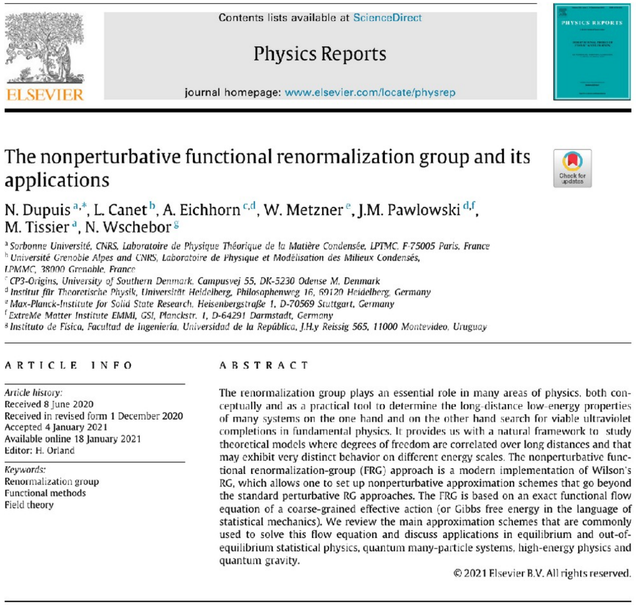
This work is dedicated to the 60th birthday of Franz Wegner

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The nonperturbative functional renormalization group and its applications

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ABSTRACT

The renormalization group plays an essential role in many areas of physics, both conceptually and as a practical tool to determine the long-distance low-energy properties of many systems on the one hand and on the other hand search for viable ultraviolet completions in fundamental physics. It provides us with a natural framework to study theoretical models where degrees of freedom are correlated over long distances and that may exhibit very distinct behavior on different energy scales. The nonperturbative functional renormalization-group (FRG) approach is a modern implementation of Wilson's RG, which allows one to set up nonperturbative approximation schemes that go beyond the standard perturbative RG approaches. The FRG is based on an exact functional flow equation of a coarse-grained effective action (or Gibbs free energy in the language of statistical mechanics). We review the main approximation schemes that are commonly used to solve this flow equation and discuss applications in equilibrium and out-of-equilibrium statistical physics, quantum many-particle systems, high-energy physics and quantum gravity.

J. Berges, N. Tetradis, and C. Wetterich, Physics Reports **363**, 223 (2002).

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N. Dupuis *et al.*, Phys. Rep. **910**, 1 (2021).

Lecture 1

1. The **renormalisation group**.
2. **Functional integration** for **ultracold quantum gases**.
3. The **effective action**.
4. The **functional renormalisation group**.
5. Summary.

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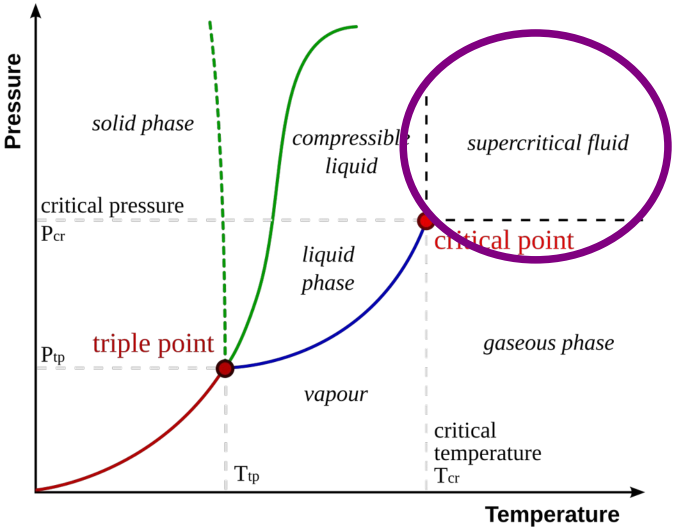
The renormalisation group

- The **renormalisation group** (RG) is a tool used to determine how a physical system changes across **physical scales**.
- It is useful in systems where the **physics** at **different scales** **becomes important**.



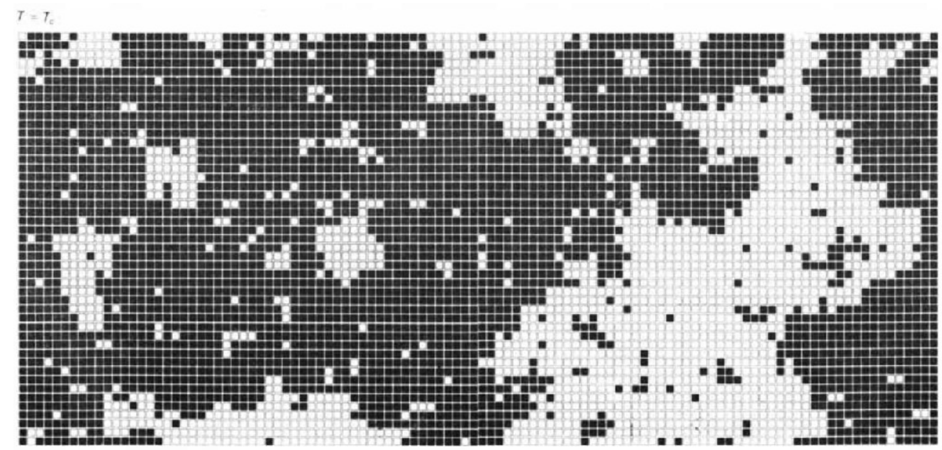
The physics of **large bodies of water** can be treated independently of the **molecular structure**. Thus, both **physical scales are decoupled**.

In contrast, in a **supercritical fluid all scales become important**. Indeed, gas bubbles and liquid drops of all sizes appear, from single molecules to macroscopic structures.

The renormalisation group

- The seminal example of the **renormalisation group** (RG) is the **ferromagnet**.

At **high temperatures**, the **orientation** of the magnetic dipoles is essentially **random**. There is **no long-range order**.

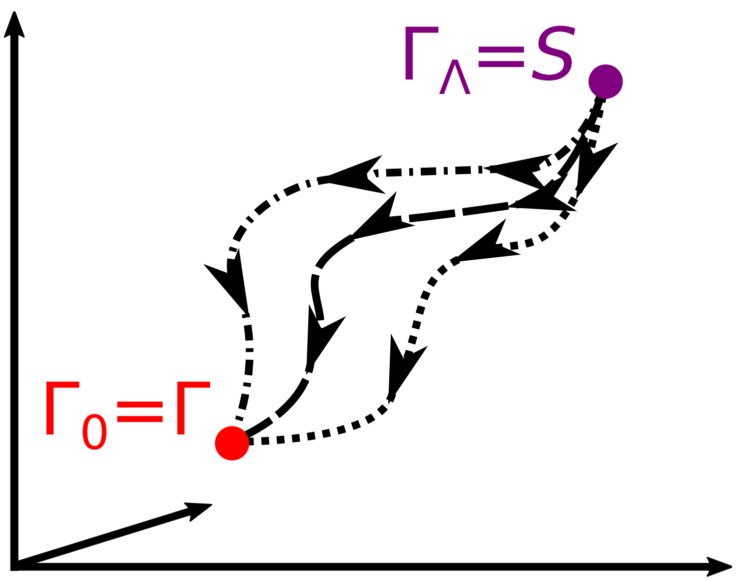


Below the critical temperature, **infinite regions** of **aligned dipoles** appear (**long-range order**). However, **fluctuations at smaller scales** are still **important**. Thus, **all scales must be considered**.

Illustration taken from K. G. Wilson, *Problems in physics with many scales of length*, Scientific American **241**, 158 (1979).

The renormalisation group

- The **RG** enables us to smoothly interpolate between **microscopic physics** (short scales) and **macroscopic physics** (long scales).
- We can use it to build an **effective theory** for long-distance physics.



The renormalisation group

- The interpolation is done through a **coarse-graining** procedure.
- This is formally implemented through RG **flow equations**.
- These equations exploit **symmetries** to examine how the physical couplings change at different scales.

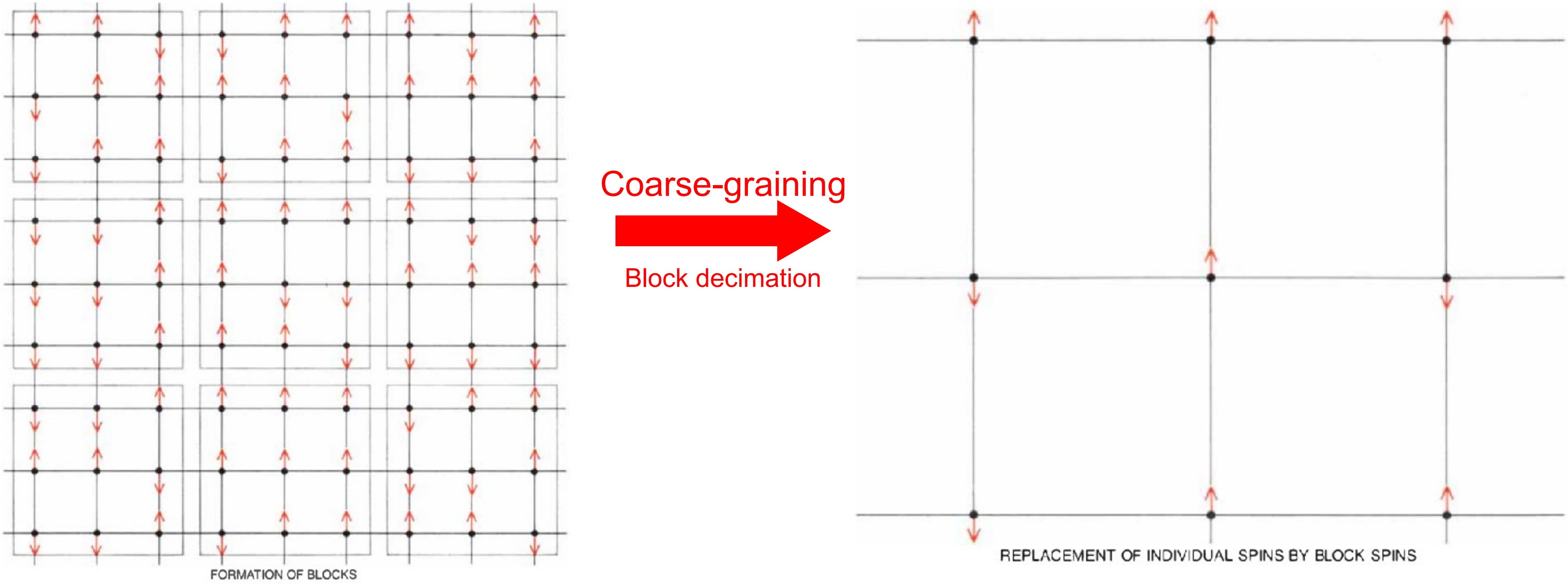


Illustration taken from K. G. Wilson, *Problems in physics with many scales of length*, Scientific American **241**, 158 (1979).

The renormalisation group (a bit of history)

- **1947 – 1949**:
  Renormalisation is born. Corrections were introduced to treat **divergences** in **field-theory** (H. Bethe, J. Schwinger, R. Feynman, S. Tomonaga, etc).

- **1953:**
  Idea that **renormalisation** in quantum field theory shows a "**group**" **of transformations** (E. Stueckelberg, A. Petermann).
  E. Stueckelberg, A. Petermann, Helv. Phys. Acta **26**, 499 (1953).

- **1954**:
  Notion of **scale-dependent couplings** for **QED**. The **renormalisation group (RG)** is born (M. Gell-Mann, F. E. Low).
  M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

- **1966:**
  Introduction of **block-spins** to connect short and long distance physics (L. P. Kadanoff).
  L.P. Kadanof, Physics **2**, 263 (1966).

- **1970:**
  Reformulation of the **RG** for **particle physics** based on the breaking of scale symmetry (C. G. Callan, K. Symanzik).
  C. G. Callan, Phys. Rev. D **2**, 1541 (1970). K. Symanzik, Commun. Math. Phys. **18**, 227 (1970).

The renormalisation group (a bit of history)

- **1971 – 1975**:
  K. Wilson developed its **modern formulation** of the **RG**, including its application to **critical phenomena**, the **momentum-shell** integration, the ε-expansion, etc. Wilson was awarded the Nobel Prize in 1982.
  K. Wilson Phys. Rev. B **4**, 3174 (1971). K. Wilson, Phys. Rev. B **4**, 3184 (1971). K.G. Wilson and M.E. Fisher, Phys. Rev. Lett. **28**, 240 (1972). K. Wilson, Rev. Mod. Phys. **47**, 773 (1975).

- **70's – 80's**:
  Extensive use of **perturbative RG** in high-energy physics, condensed matter, and statistical physics.

- **1984**:
  J. Polchinski develops its **exact RG** formulation, including irrelevant couplings.
  J. Polchinski, Nucl. Phys. B. **231**, 269 (1984).

- **1993 – 1994:**
  C. Wetterich, U. Ellwanger, T. Morris independently develop a new **exact** formulation of the **RG** based on a **Legendre-transformed effective action**. This would become the **FRG** approach studied in these lectures.
  C. Wetterich, Phys. Lett. B. **301**, 90 (1993). U. Ellwanger, Z. Phys. C **62**, 503 (1994). T. Morris. Internat. J. Modern Phys. A **09**, 2411 (1994).

The functional renormalisation group

- The **FRG** is a **non-perturbative formulation** of **Wilson's RG**.
- Essentially, it enables us to extract the **free energy** and **correlation functions**.
- It is particularly useful for studying **strongly-correlated systems** and **criticality**.
- It has been applied in **statistical physics**, **high-energy physics**, and even for studying **quantum gravity**.

The functional renormalisation group

- While somewhat niche, the FRG has applied to a variety of problems relevant to **ultracold atomic physics**, such as
  - One-component **Bose gases**,
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  - **Fermi gases**: BCS-BEC crossover,
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  - Bose gases in **optical lattices**,
    A. Rançon and N. Dupuis, Phys. Rev. A **85**, 063607 (2012), Phys. Rev. A **86**, 043624 (2012).
  - Few bosons: **Efimov** physics,
    S. Floerchinger *et al.*, Few-Body Syst. **51**, 153 (2011). R. Schmidt and S. Moroz, Phys. Rev. A **81**, 052709 (2010).
  - Quantum **mixtures**,
    F. Isaule *et al.*, Physical Review A **103**, 013318 (2021). von Milczewski *et al.*, Phys. Rev. A **105**, 013317 (2022). B. Krippa, arXiv:2606.16029.
  - Fermi and Bose **polarons**.
    R. Schmidt and T. Enss, Phys. Rev. A **83**, 063620 (2011). F. Isaule *et al.*, Phys. Rev. A **104**, 023317 (2021).

The functional renormalisation group

- Why is the RG (and thus the FRG) relevant for **many-body calculations**?
- It enables us to smoothly take **fluctuations**, both quantum and thermal, **into account** across the relevant **physical scales**:
  - thermal scale: **de Broglie wavelength**,

$$\lambda_{\mathrm{th}} = \sqrt{\frac{2\pi\hbar^2}{Mk_BT}}.$$

  - inter-particle distance: **chemical potential**,

$$n^{1/d}.$$

  - interaction scales: **scattering length**, **effective range**,

$$a,\, r_{\mathrm{eff}}.$$

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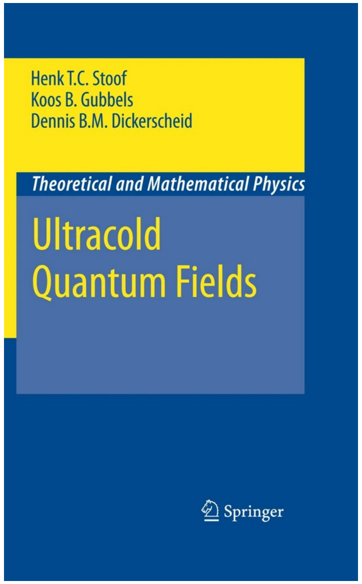
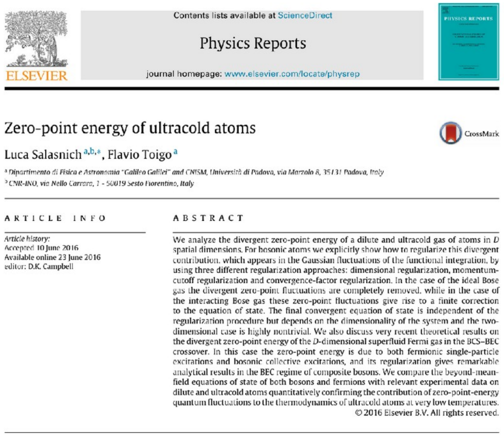
Functional integration

- Before introducing the **effective action** and the **FRG**, we will revise the **field-theory formalism** for **quantum gases**.
- In the following, we will examine a **weakly-interacting Bose gas**.
- We will revise conventions and results that will be used throughout these lectures.





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Weakly-interacting Bose gas

- Let's consider a d-dimensional and **uniform** gas of **bosons** that **interact** through **weak contact repulsive interactions**.
- The **Hamiltonian** in **second quantisation**:

$$\hat{H} = \int d\boldsymbol{x} \left(\frac{1}{2M} \nabla \hat{\Phi}^\dagger(\boldsymbol{x}) \nabla \hat{\Phi}(\boldsymbol{x}) \right) + \frac{g}{2} \int d\boldsymbol{x}\, \hat{\Phi}^\dagger(\boldsymbol{x}) \hat{\Phi}^\dagger(\boldsymbol{x}) \hat{\Phi}(\boldsymbol{x}) \hat{\Phi}(\boldsymbol{x}).$$

$\hat{\psi}, \hat{\psi}^\dagger$: Annihilator and creator **operators**.
M: Mass.
g: Contact interaction strength.

- The **grand-canonical partition function**:

$$\mathcal{Z} = \mathrm{tr}\left[e^{-\beta(\hat{H} - \mu \hat{N})} \right].$$

μ: Chemical potential.
$\beta = 1/T$
$\hat{N} = \int d\boldsymbol{x}\, \hat{\Phi}^\dagger(\boldsymbol{x}) \hat{\Phi}(\boldsymbol{x})$

- Note that, from now on, we use **natural units**, $\hbar = k_B = 1$.

L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, Oxford, 2016).

Weakly-interacting Bose gas

- These Bose gases can form a **condensate** in **two** and **three dimensions**,

$$\Phi = \langle \hat{\Phi} \rangle.$$

Wave function of the condensate (**order parameter**)

- The formation of a condensate is tied to the onset of **long-range order** and **spontaneous symmetry breaking**.
- These aspects appear naturally within the **FRG**.
- In the following, we will revise the **path-integral** formulation of the weakly-interacting Bose gas.

Path-integral formalism

- From now on, we will work within a **field-theory** formulation, with **bosonic fields** ϕ and $\phi^\dagger$.
- These are introduced through **coherent states** $\hat{\Phi}|\phi\rangle = \phi|\phi\rangle$.
- One obtains the **microscopic action**,

$$\mathcal{S}[\boldsymbol{\phi}] = \int_x \left[\phi^\dagger(x)\left(\partial_\tau - \frac{\nabla^2}{2M} - \mu\right)\phi(x) + \frac{g}{2}(\phi^\dagger(x)\phi(x))^2\right],$$

$\boldsymbol{\phi} = (\phi, \phi^\dagger)$

where $x=(\tau,\boldsymbol{x})$, with $\tau=i\,t$ the **imaginary time**. The integral,

$$\int_x = \int_0^\beta d\tau \int d\boldsymbol{x}.$$

- The model shows a **U(1)-symmetry**,

$$\phi \longrightarrow \phi e^{i\alpha}, \qquad \phi^\dagger \longrightarrow \phi^\dagger e^{-i\alpha}.$$

Path-integral formalism

- The **grand-canonical partition function** takes the form of a **path integral**,

$$\mathcal{Z} = \int_{\phi(\beta)=\phi(0)} \mathcal{D}\phi e^{-\mathcal{S}[\phi]}.$$

- We can then extract the thermodynamics from the **grand-canonical potential**,

$$\Omega = -\beta^{-1} \log \mathcal{Z},$$

$$d\Omega = -P d\mathcal{V} - S dT - N d\mu.$$

- To compute the partition function, we need to **integrate all the different paths**.

Mean-field solution

- We start by examining the **mean-field solution** (MF), where we impose $\phi(x)=\phi^\dagger(x)=\phi_0$. The **action** takes the form

$$\mathcal{S}_0 = \beta\mathcal{V}\left(-\mu\phi_0^2 + \frac{g}{2}\phi_0^4\right).$$

$$\int_0^\beta d\tau = \beta$$
$$\int d\boldsymbol{x} = \mathcal{V}$$

- We only need to consider the **classical path**,

$$\mathcal{Z}_0 = e^{-\mathcal{S}_0} = e^{-\beta\Omega_0} \quad \longrightarrow \quad \frac{\Omega_0}{\mathcal{V}} = -\mu\phi_0^2 + \frac{g}{2}\phi_0^4.$$

- The **condensate** is captured by ϕ_0.

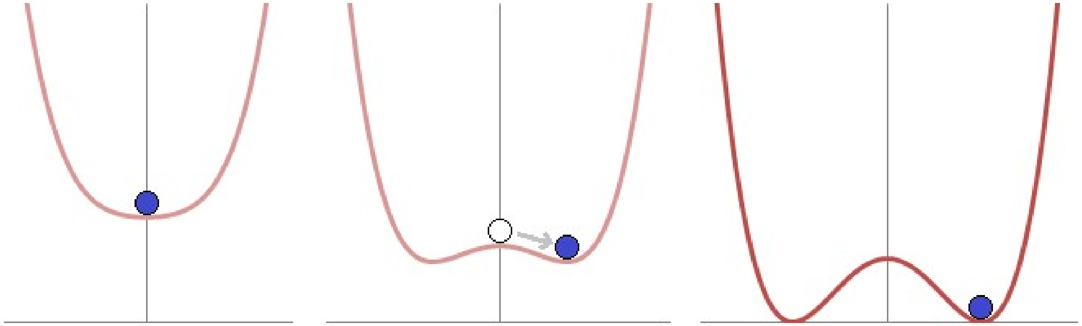
Mean-field solution

- From the **saddle-point condition**, we obtain

$$\left(\frac{\partial \Omega_0}{\partial \phi_0}\right) = 0 \quad \longrightarrow \quad \rho_0 = \phi_0^2 = \mu/g.$$

where ρ_0 is the MF **condensate density**.

- This **condition** reflects the **breaking** of the **U(1)-symmetry**.



- The density ρ_0 is seen as the **order parameter**.

Mean-field solution

- The **grand-canonical potential** now reads

$$\frac{\Omega_0}{\mathcal{V}} = -\frac{\mu}{2g}.$$

- If we extract the **density**

$$n_0 = -\frac{1}{\mathcal{V}}\left(\frac{\partial \Omega_0}{\partial \mu}\right) \quad \longrightarrow \quad n_0 = \mu/g = \rho_0.$$

- Thus, at the **MF level**, the condensate and number density are equal (**no condensate depletion**).

- Also note that at the **MF level**, the **dimensionality** does **not play a role**.

Gaussian solution

- To take **fluctuations** into account, we need to go **beyond MF**.
- We can **expand the fields** around the condensate as

$$\phi(x) = \phi_0 + \eta(x).$$

- At the **Gaussian** level, we keep up to the **quadratic terms** in the fields. The **action** takes the form

$$\mathcal{S}[\boldsymbol{\eta}] = \underbrace{\boxed{\mathcal{S}_0}}_{\text{MF action}} + \underbrace{\boxed{\mathcal{S}_2[\boldsymbol{\eta}]}}_{\text{Gaussian action}},$$

where

$$\mathcal{S}_2[\boldsymbol{\eta}] = \int_x \Big[\eta^\dagger(x) \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu \right) \eta(x) \\ + \frac{g}{2}\phi_0^2 \left(\eta^\dagger(x)\eta^\dagger(x) + 2\eta^\dagger(x)\eta(x) + 2\eta(x)\eta^\dagger(x) + \eta(x)\eta(x) \right) \Big].$$

Gaussian solution

- In **momentum space**,

$$\phi(x) = \int_q e^{i(\omega_n \tau + \boldsymbol{q}\cdot\boldsymbol{x})} \phi(q),$$

where $q{=}(\omega_n,\boldsymbol{q})$, with the $\omega_n{=}2\pi n T$ bosonic **Matsubara frequencies**. The integral reads

$$\int_q = T \sum_{n=-\infty}^{\infty} \int_{\boldsymbol{q}}, \qquad \int_{\boldsymbol{q}} = \frac{\mathcal{V}}{(2\pi)^d} S_d \int_0^\infty q^{d-1} dq,$$

where $S_d{=}2\pi^{d/2}/\Gamma(d/2)$, with Γ the gamma function.

- The sum over n is called a **Matsubara sum**. At **zero temperature**, it takes the expected form

$$T \sum_n \longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega.$$

Gaussian solution

- The **Gaussian action** in **momentum space** can be written as

$$\mathcal{S}_2[\boldsymbol{\eta}] = \frac{1}{2}\int_q \left(\eta^\dagger(q), \eta(-q)\right) G^{-1}(q) \begin{pmatrix} \eta(q) \\ \eta^\dagger(-q) \end{pmatrix},$$

where

$$G^{-1}(q) = \begin{pmatrix} -i\omega_n + \frac{q^2}{2M} - \mu + 2g\phi_0^2 & g\phi_0^2 \\ g\phi_0^2 & i\omega_n + \frac{q^2}{2M} - \mu + 2g\phi_0^2 \end{pmatrix}.$$

- Within the **Bogoliubov approximation**, we use the MF solution for the condensate $\phi_0^2 = \mu/g$. Thus,

$$G^{-1}(q) = \begin{pmatrix} -i\omega_n + \frac{q^2}{2M} + \mu & \mu \\ \mu & i\omega_n + \frac{q^2}{2M} + \mu \end{pmatrix}.$$

Gaussian solution

- The **path integral** of this action is a **Gaussian integral**. One obtains

$$\Omega_2 = \frac{1}{2}\int_q \log\det(G^{-1}) = \int_q \log[\beta^2(\omega_n^2 + E_{\boldsymbol{q}}^2)],$$

where

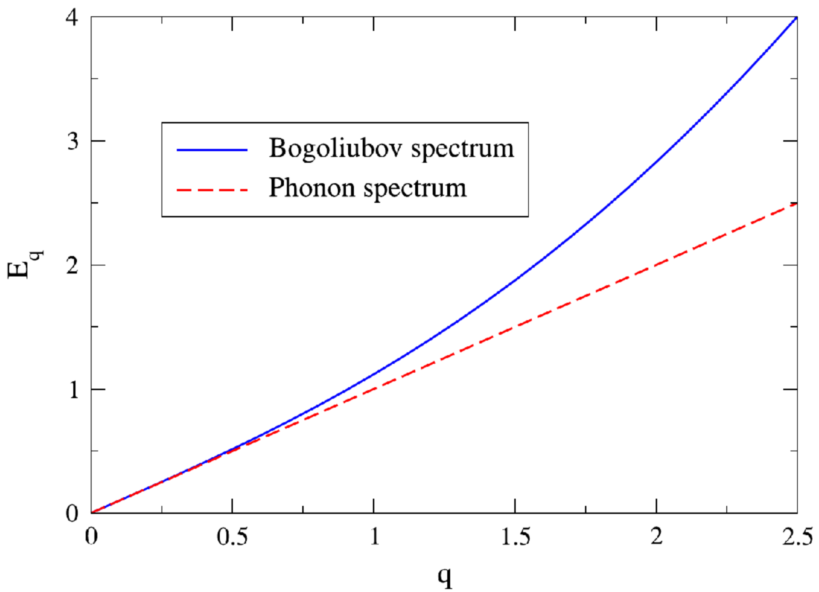
$$E_{\boldsymbol{q}} = \sqrt{\frac{\boldsymbol{q}^2}{2M}\left(\frac{\boldsymbol{q}^2}{2M} + 2\mu\right)},$$

is the known **Bogoliubov spectrum**.

- At **low momenta**, it shows a **phonon spectrum**

$$E_{\boldsymbol{q}} \sim c_B|\boldsymbol{q}|.$$

$c_B = \sqrt{\mu/M}$: sound velocity



$$\text{Healing scale} : p_h = \sqrt{2M\mu}.$$

Figure taken from L. Salasnich and F. Toigo, Phys. Rep. **640**, 1 (2016).

Gaussian solution

- After solving the **Matsubara sum**, we obtain

$$\Omega_2 = \underbrace{\frac{1}{2}\int_{\boldsymbol{q}} E_{\boldsymbol{q}}}_{\Omega_2^{(T=0)}} + \underbrace{\beta^{-1}\int_{\boldsymbol{q}} \log\left(1 - e^{-\beta E_{\boldsymbol{q}}}\right)}_{\Omega_2^{(T>0)}} .$$

- The full **grand-canonical potential** is

$$\Omega = \Omega_0 + \Omega_2^{(T=0)} + \Omega_2^{(T>0)},$$

  where Ω_0 is the **MF solution**.

- The zero-temperature Gaussian term is **UV divergent**.
- This is due to the use of **contact interactions**, where particles interact with zero distance (infinite momentum).

Gaussian solution

- The integral needs to be **regularised**.
- We recall that the **two-body** scattering is captured by the T-**matrix.**
- In **three dimensions** and at zero energy, it reads

$$T^{(2\mathrm{B})} = \frac{4\pi a}{M},$$

where a is the ***s*-wave scattering length**.

- By using the **Lippmann–Schwinger equation**,

$$\frac{1}{T^{(2\mathrm{B})}} = \frac{1}{g} + \frac{1}{2}\int_{\boldsymbol{q}} \frac{1}{\epsilon_{\boldsymbol{q}}}, \qquad \epsilon_q = \frac{q^2}{2M}$$

which enables us to regularise the previous integrals.

Gaussian solution

- We can obtain the **Lee-Huang-Yang (LHY)** corrections for the **zero-temperature** gas in **three dimensions**,

$$\mu = \frac{4\pi a n}{M}\left(\underbrace{1}_{\text{MF}} + \underbrace{\frac{32}{3\pi^{1/2}}(na^3)^{1/2}}_{\text{LHY}}\right),$$

$$E/N = \frac{2\pi a n}{M}\left(1 + \underbrace{\frac{128}{15\pi^{1/2}}(na^3)^{1/2}}_{\text{LHY}}\right),$$

$$\frac{\rho_0}{n} = 1 - \frac{8}{3\pi^{1/2}}(na^3)^{1/2}.$$

condensate depletion

Gaussian solution

- The **LHY** corrections give a **good** description at very **low temperatures**. However, it fails at higher temperatures.
- One problem of the **Bogoliubov** approximation is that it is **not consistent**: it uses the **MF condensate density**.
- This can be **improved** with **Popov's approach**.
- However, to **describe** the **superfluid phase transition**, one needs to rely on **more sophisticated approaches**, such as QMC and the RG.

Lecture 1

1. The **renormalisation group**.
2. **Functional integration** for **ultracold quantum gases**.
3. The **effective action**.
4. The **functional renormalisation group**.
5. Summary.

The effective action

- It can be more convenient to work in terms of an **effective action** $\Gamma[\varphi]$ that has already incorporated **fluctuations**
- This means that it is a **classical action**, fulfilling the **principle of least action**.
- The effective action contains all the relevant properties of a field theory.
  - It is the **generator** of the **one-particle irreducible** (1PI) **correlation functions**.
  - It naturally implements **spontaneous symmetry breaking** by **minimising** the **potential** component of Γ.

The effective action

- To derive the effective action, we first define the **generating functional**

$$\mathcal{Z}[\boldsymbol{J}] = e^{W[\boldsymbol{J}]} = \int \mathcal{D}\phi e^{-\mathcal{S}[\phi]+\int_x \boldsymbol{J}\cdot\boldsymbol{\phi}},$$

  where J are **source fields**.

- The n-point **correlation functions** are given by

$$G^{(n)}(x_1, ..., x_n) = \langle \boldsymbol{\phi}(x_1)...\boldsymbol{\phi}(x_n) \rangle = \left. \frac{\delta^n W[\boldsymbol{J}]}{\delta \boldsymbol{J}(x_1)...\delta \boldsymbol{J}(x_n)} \right|_{\boldsymbol{J}=0}.$$

- The correlation functions are essential for studying the onset of **long-range order** and **phase transitions**.

The effective action

- We also define **classical fields** as

$$\varphi = \langle \phi(x) \rangle = \frac{\delta W[\boldsymbol{J}]}{\delta \boldsymbol{J}(x)}\bigg|_{\boldsymbol{J}=0}.$$

- The **effective action** is defined by means of a **Legendre transformation**

$$\Gamma[\varphi] = -W[\boldsymbol{J}] + \int_x \boldsymbol{J} \cdot \varphi.$$

- It **only depends** on the **classical fields**, as it is independent of the source fields,

$$\frac{\Gamma[\varphi]}{\delta \boldsymbol{J}} = -\frac{\delta W[\boldsymbol{J}]}{\delta \boldsymbol{J}} + \varphi = 0.$$

The effective action

- By inverting the classical field,

$$\frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}(x)} = \boldsymbol{J}(x) \qquad \xrightarrow{\boldsymbol{J}=0} \qquad \frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}(x)} = 0,$$

  which is the **equation of motion** (principle of least action).

- Essentially, this equation is the **saddle-point condition**, from which we can obtain $\boldsymbol{\varphi}_0$.

- The **grand-canonical potential** is simply

$$\Omega = \beta^{-1}\Gamma[\boldsymbol{\varphi}_0].$$

- In turn, the **correlation functions** are

$$G^{(n)}(x_1, ..., x_n) = \langle\boldsymbol{\phi}(x_1)...\boldsymbol{\phi}(x_n)\rangle = \left.\frac{\delta^n\Gamma[\boldsymbol{\varphi}]}{\delta\boldsymbol{\varphi}(x_1)...\delta\boldsymbol{\varphi}(x_n)}\right|_{\boldsymbol{J}=0}.$$

The effective action

- The **effective action** can be computed from a **perturbative expansion**,

$$\Gamma[\varphi] = \underbrace{\boxed{\mathcal{S}[\varphi]}}_{\text{MF}} + \underbrace{\boxed{\frac{1}{2}\mathrm{tr}\log\left(\mathcal{S}^{(2)}[\varphi]\right) + \ldots}}_{\text{Fluctuations}} ,$$

  where

$$\mathcal{S}^{(n)}[\varphi] = \left.\frac{\delta^n \mathcal{S}[\varphi]}{\delta\varphi(x_1)\ldots\varphi(x_n)}\right|_{\varphi=\varphi_0} .$$

- In the following, we will study how to compute Γ from the **FRG**.

Lecture 1

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Functional renormalisation group

- Within the FRG, the **effective action** is obtained by solving a **flow equation**.
  C. Wetterich, Phys. Lett. B **301**, 90 (1993).

- We work in terms of a **scale-dependent effective action** Γ_k that depends on a **momentum scale** k.

- This momentum scale **suppresses fluctuations** $|q| \lesssim k$. Therefore,

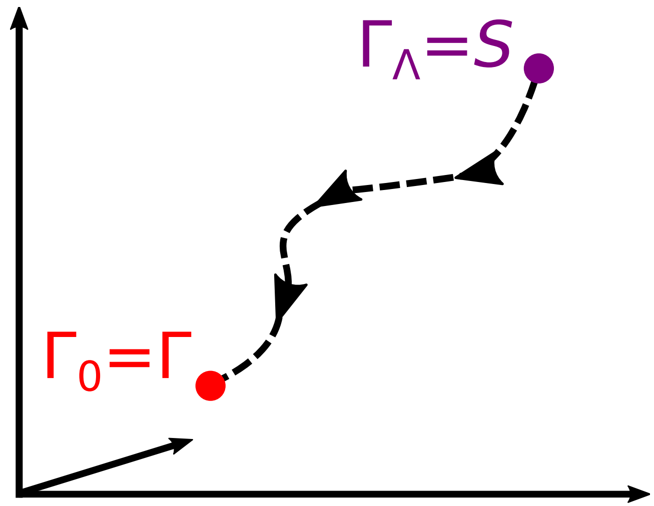
$$\boxed{\Gamma_{\Lambda \gg 0} = \mathcal{S},} \qquad \boxed{\Gamma_{k=0} = \Gamma.}$$

Known UV physics (microscopic) — Physical IR solution (macroscopic)

- Γ_k is sometimes called the **effective average action**.

Functional renormalisation group

- Thus, Γ_k smoothly connects the known **microscopic action** (UV) with the physical **effective action** Γ (IR).



- The flow of Γ_k is dictated by the **Wetterich equation**
  C. Wetterich, Phys. Lett. B **301**, 90 (1993).

$$\boxed{\partial_k \Gamma_k = \frac{1}{2} \mathrm{tr} \left[\partial_k \mathcal{R}_k (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right] .}$$

The scale-dependent action

- To define, we add a **scale-dependent** quadratic **mass term** to the microscopic action,

$$\mathcal{S}_k[\phi] = \mathcal{S}[\phi] + \Delta\mathcal{S}_k[\phi],$$

where

$$\Delta\mathcal{S}_k = \frac{1}{2}\int_q \boldsymbol{\phi}^\dagger(q)\mathcal{R}_k(q)\boldsymbol{\phi}(q),$$

and R_k is called the **regulator** function.

- Therefore, we now work in terms of a scale-dependent **generating functional**

$$\mathcal{Z}_k[\boldsymbol{J}] = e^{W_k[\boldsymbol{J}]} = \int \mathcal{D}\boldsymbol{\phi} e^{-\mathcal{S}[\boldsymbol{\phi}]-\Delta\mathcal{S}_k[\boldsymbol{\phi}]+\int_x \boldsymbol{J}\cdot\boldsymbol{\phi}}.$$

The regulator

- The **regulator** is chosen such that,

$$\Gamma_\Lambda = \mathcal{S}, \qquad \Gamma_0 = \Gamma.$$

- It must fulfils

$$
\begin{aligned}
R_k(q) &\longrightarrow \infty, & (k \to \Lambda) \\
R_k(q) &\longrightarrow 0, & (k \to 0) \\
R_k(q) &\propto q^2. & (q \to 0)
\end{aligned}
$$

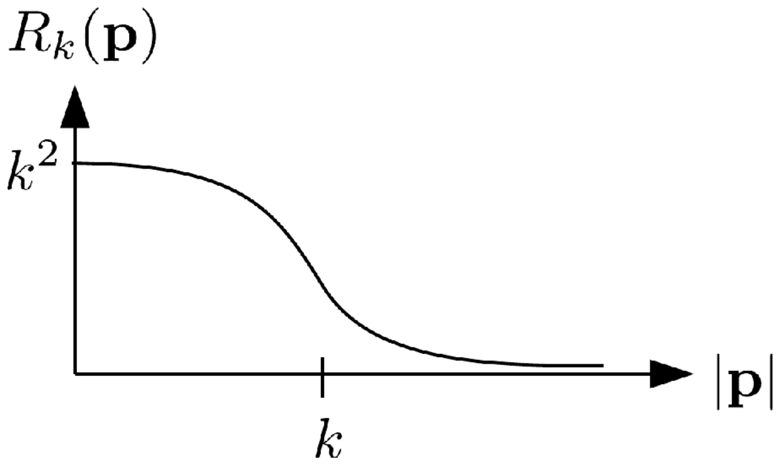


Illustration of a typical regulator. Figure taken from N. Dupuis *et al.*, Phys. Rep. **910**, 1 (2021).

- Overall, it must suppress fluctuations for lower momenta $|q| \lesssim k$, but leave larger momenta unchanged.

The scale-dependent effective action

- The **scale-dependent effective action** is defined as

$$\Gamma_k[\boldsymbol{\varphi}] = -W_k[\boldsymbol{J}] + \int_x \boldsymbol{J} \cdot \boldsymbol{\varphi} - \Delta\mathcal{S}_k[\boldsymbol{\varphi}].$$

- It is easy to see that for $k \to 0$, where $R_k \to 0$, we have that

$$\Gamma_{k\to 0}[\boldsymbol{\varphi}] = \Gamma[\boldsymbol{\varphi}],$$

which is the **physical effective action**, and the solution to the problem.

- We will now briefly derive the flow equation for Γ_k.

The Wetterich equation

- We start from the scale-dependent partition function

$$\mathcal{Z}_k[\boldsymbol{J}] = e^{W_k[\boldsymbol{J}]} = \int \mathcal{D}\boldsymbol{\phi} e^{-\mathcal{S}[\boldsymbol{\phi}]-\Delta\mathcal{S}_k[\boldsymbol{\phi}]+\int_x \boldsymbol{J}\cdot\boldsymbol{\phi}}.$$

- We have that

$$\Delta\mathcal{S}_k = \frac{1}{2}\int_q \phi^\dagger(-q)R_k(q)\phi(q),$$

$$\partial_k\mathcal{Z}_k = \boxed{\partial_k W_k}\,\boxed{e^{W_k}} = -\frac{1}{2}\int \mathcal{D}\boldsymbol{\phi}\boxed{(\boldsymbol{\phi}^\dagger\partial_k\mathcal{R}_k\boldsymbol{\phi})}e^{-\mathcal{S}-\Delta\mathcal{S}_k+\int_x \boldsymbol{J}\cdot\boldsymbol{\phi}}$$

$$= -\frac{1}{2}(\partial^\dagger_{\boldsymbol{J}}\,\partial_k\mathcal{R}_k\,\partial_{\boldsymbol{J}})e^{W_k} \longleftarrow \quad \varphi_{\boldsymbol{J}} = \frac{\delta W[\boldsymbol{J}]}{\delta \boldsymbol{J}(x)}$$

$$= -\frac{1}{2}(\boldsymbol{\varphi}^\dagger\,\partial_k\mathcal{R}_k\,\boldsymbol{\varphi} + \mathrm{tr}(\partial_k\mathcal{R}_k\,\delta_{\boldsymbol{J}}\boldsymbol{\varphi}))\boxed{e^{W_k}}$$

It cancels with the same term on the LHS

- Here tr denotes both a trace and an integral over momenta $\int_q$.

The Wetterich equation

- To continue, we need one equation for $\partial_k W_k$ and one equation for $\partial_{\boldsymbol{J}}\boldsymbol{\varphi}$. We employ the definition of Γ_k,

$$\Gamma_k[\boldsymbol{\varphi}] = -W_k[\boldsymbol{J}] + \int_x \boldsymbol{J}\cdot\boldsymbol{\varphi} - \Delta\mathcal{S}_k[\boldsymbol{\varphi}].$$

- By differentiating with respect to k,

$$\partial_k W_k = -\partial_k \Gamma_k - \frac{1}{2}\boldsymbol{\varphi}^\dagger\,\partial_k \mathcal{R}_k\,\boldsymbol{\varphi}.$$

- While by taking two functional derivatives with respect to $\boldsymbol{\varphi}$,

$$\Gamma_k^{(2)} = \frac{\delta \boldsymbol{J}}{\delta\boldsymbol{\varphi}} - \mathcal{R}_k \qquad \longrightarrow \qquad \partial_{\boldsymbol{J}}\boldsymbol{\varphi} = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1},$$

where

$$\Gamma_k^{(2)}(q) = \frac{\delta^2\Gamma_k}{\delta\boldsymbol{\varphi}(-q)\delta\boldsymbol{\varphi}(q)}.$$

The Wetterich equation

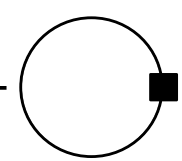
- By replacing everything, we obtain the **Wetterich equation**

$$\boxed{\partial_k \Gamma_k = \frac{1}{2} \mathrm{tr} \left[\partial_k \mathcal{R}_k (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right].}$$

- Importantly, as mentioned, tr denotes both a trace and an integral over momenta $\int_q$. Thus, more formally

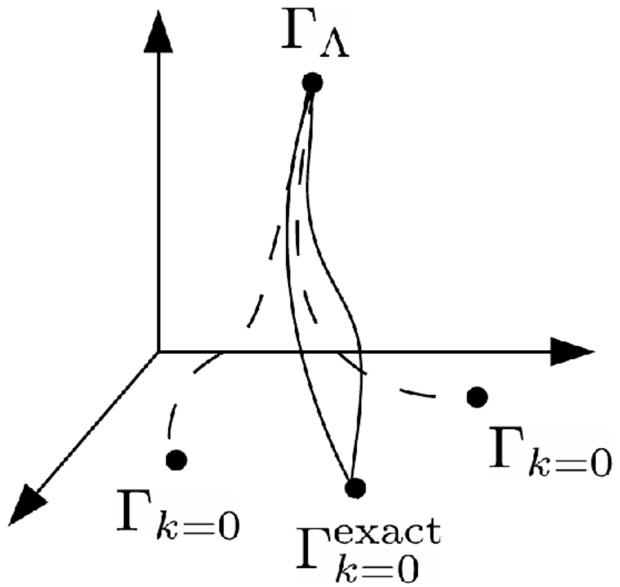
$$\partial_k \Gamma_k = \frac{1}{2} \int_q \mathrm{tr} \left[\underbrace{\partial_k \mathcal{R}_k(q)}_{\text{Insertion}} \underbrace{\left(\Gamma_k^{(2)}(q) + \mathcal{R}_k(q)\right)^{-1}}_{\text{Propagator } G_k} \right].$$

- It has a **one-loop** structure.

$\partial_k \Gamma_k = \frac{1}{2}$ 

The Wetterich equation

- The **Wetterich equation** is **exact.**
- Because of this, the FRG was initially called the **exact renormalisation group** (ERG).
- However, in practice, the Wetterich equation cannot be solved exactly, and so one needs to rely on **approximations**.



In practical application one obtains an approximate solution. Figure taken from N. Dupuis *et al.*, Phys. Rep. **910**, 1 (2021).

- Nevertheless, the FRG is usually **non-perturbative**.
- Thus, another common name for the FRG is the **non-perturbative renormalisation group** (NPRG).

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Summary

- The **RG** can be used to study **many-body systems**.
- The **FRG** is a **non-perturbative** RG formulation that enables us to systematically study field-theory models.
- We have introduced the **field-theory** formulation of the **Bose gas**.
- In the next lecture, we will apply the **FRG** to study **Bose gases**.
- Next lecture:
  - → FRG for the O(2)-model.