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Functional renormalisation group for quantum many-body systems

Lecture 2:

Functional renormalisation group for the O(2)-model

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Lecture 2

1. Bose gases and the **O(2)-model**.
2. The **FRG for the O(2)-model**. LPA and the **derivative expansion**.
3. **Beyond LPA** and **critical** behaviour.
4. Summary.

Critical exponents from the effective average action

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Abstract

We compute the critical behaviour of three-dimensional scalar theories using a new exact non-perturbative evolution equation. Our values for the critical exponents and other universal constants agree well with previous precision estimates. The evolution equation describes the scale dependence of the effective average action (coarse-grained effective action). It is not restricted to critical phenomena and can be used in the presence of massless (Goldstone) modes dealing successfully with the infrared problems. Our results establish the viability of our method for precise computations in a non-perturbative context.

N. Tetradis and C. Wetterich, Nucl. Phys. B **422**, 541 (1994).

Critical exponents from optimised renormalisation group flows

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Received 5 March 2002; accepted 8 March 2002

Abstract

Within the exact renormalisation group, the scaling solutions for $O(N)$ symmetric scalar field theories are studied to leading order in the derivative expansion. The Gaussian fixed point is examined for $d > 2$ dimensions and arbitrary infrared regularisation. The Wilson–Fisher fixed point in $d = 3$ is studied using an optimised flow. We compute critical exponents and subleading corrections-to-scaling to high accuracy from the eigenvalues of the stability matrix at criticality for all N. We establish that the optimisation is responsible for the rapid convergence of the flow and polynomial truncations thereof. The scheme dependence of the leading critical exponent is analysed. For all $N \geqslant 0$, it is found that the leading exponent is bounded. The upper boundary is achieved for a Callan–Symanzik flow and corresponds, for all N, to the large-N limit. The lower boundary is achieved by the optimised flow and is closest to the physical value. We show the reliability of polynomial approximations, even to low orders, if they are accompanied by an appropriate choice for the regulator. Possible applications to other theories are outlined. © 2002 Published by Elsevier Science B.V.

PACS: 11.10.Hi; 11.15.Tk; 64.60.Ak; 11.10.Kk

D. Litim, Nucl. Phys. B **631**, 128 (2002).

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Non-perturbative renormalization flow in quantum field theory and statistical physics

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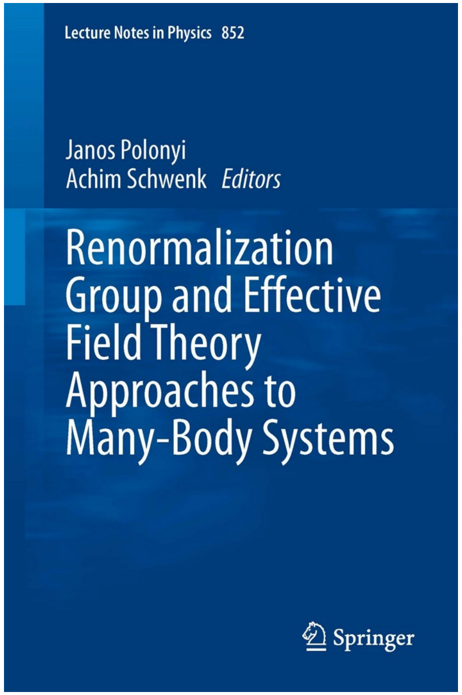
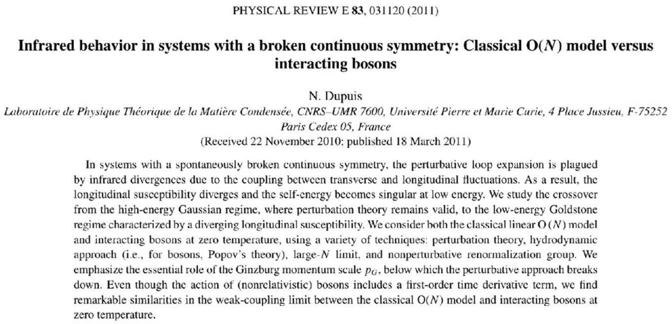
This work is dedicated to the 60th birthday of Franz Wegner

J. Berges, N. Tetradis, and C. Wetterich, Physics Reports **363**, 223 (2002).




PHYSICAL REVIEW E **83**, 031120 (2011)

Infrared behavior in systems with a broken continuous symmetry: Classical O(N) model versus interacting bosons

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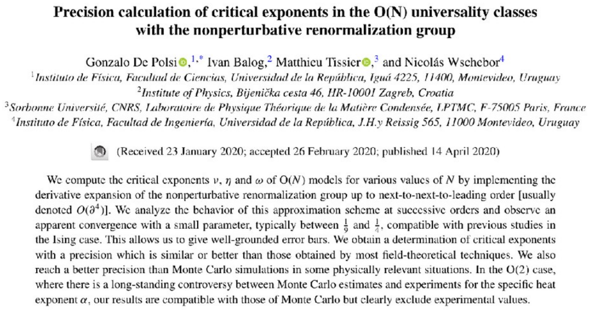
In systems with a spontaneously broken continuous symmetry, the perturbative loop expansion is plagued by infrared divergences due to the coupling between transverse and longitudinal fluctuations. As a result, the longitudinal susceptibility diverges and the self-energy becomes singular at low energy. We study the crossover from the high-energy Gaussian regime, where perturbation theory remains valid, to the low-energy Goldstone regime characterized by a diverging longitudinal susceptibility. We consider both the classical linear O (N) model and interacting bosons at zero temperature, using a variety of techniques: perturbation theory, hydrodynamic approach (i.e., for bosons, Popov's theory), large-N limit, and nonperturbative renormalization group. We emphasize the essential role of the Ginzburg momentum scale p_G, below which the perturbative approach breaks down. Even though the action of (nonrelativistic) bosons includes a first-order time derivative term, we find remarkable similarities in the weak-coupling limit between the classical O(N) model and interacting bosons at zero temperature.


Precision calculation of critical exponents in the O(N) universality classes with the nonperturbative renormalization group

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We compute the critical exponents ν, η and ω of O(N) models for various values of N by implementing the derivative expansion of the nonperturbative renormalization group up to next-to-next-to-leading order [usually denoted $O(\partial^4)$]. We analyze the behavior of this approximation scheme at successive orders and observe an apparent convergence with a small parameter, typically between $\frac{1}{9}$ and $\frac{1}{4}$, compatible with previous studies in the Ising case. This allows us to give well-grounded error bars. We obtain a determination of critical exponents with a precision which is similar or better than those obtained by most field-theoretical techniques. We also reach a better precision than Monte Carlo simulations in some physically relevant situations. In the O(2) case, where there is a long-standing controversy between Monte Carlo estimates and experiments for the specific heat exponent α, our results are compatible with those of Monte Carlo but clearly exclude experimental values.

A. Schwenk and J Polonyi (eds.), *Renormalization group and effective field theory approaches to many-body systems* (Springer, 2012).

N. Dupuis, Phys. Rev. E **83**, 031120 (2011).

G. De Polsi, I. Balog, M. Tissier, and N. Wschebor, Phys. Rev. E **101**, 042113 (2020).

Lecture 2

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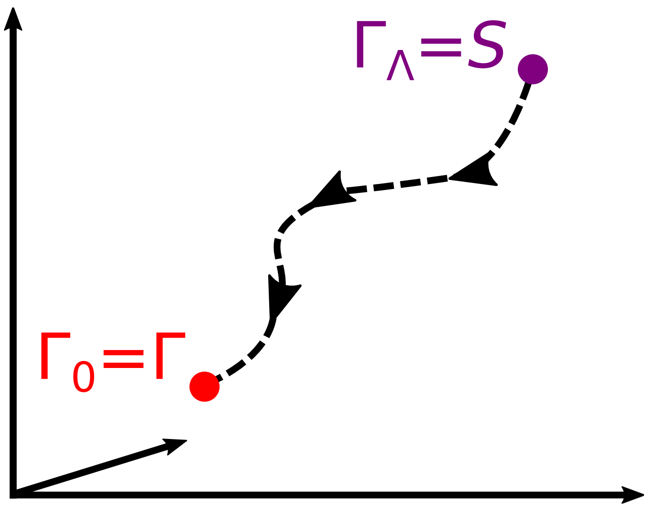
Weakly-interacting Bose gas

- We will apply the FRG to the **weakly interacting Bose gas**.
- We recall that the **microscopic action** of this system is

$$\mathcal{S}[\phi] = \int_x \left[\phi^\dagger(x)\left(\partial_\tau - \frac{\nabla^2}{2M} - \mu\right)\phi(x) + \frac{g}{2}(\phi^\dagger(x)\phi(x))^2\right].$$

- This microscopic action will be used as the starting point for solving the **Wetterich equation**.



Classical field approximation

- For simplicity, we will start examining a classical model. To do this, we will work within the **classical field approximation**.
- We write the fields as

$$\phi(\tau, \boldsymbol{x}) = \phi_{n=0}(\boldsymbol{x}) + T \sum_{n \neq 0} e^{i\omega_n \tau} \phi_n(\boldsymbol{x}).$$

- At **high temperatures** (near the phase transition), the system is dominated by fluctuations for momenta

$$|q| \lesssim \lambda_{\text{th}}^{-1}, \qquad \lambda_{\text{th}} = \sqrt{\frac{2\pi}{MT}}.$$

- In such a regime, the non-static modes $(n \neq 0)$ decouple from the static one $(n = 0)$.
- Therefore, we can neglect the terms with $n \neq 0$.

Classical field approximation

- For $n=0$, we can integrate τ directly. We obtain

$$\mathcal{S}[\boldsymbol{\phi}] = \beta \int d\boldsymbol{x} \left[\phi^\dagger(\boldsymbol{x}) \left(-\frac{\nabla^2}{2M} - \mu\right) \phi(\boldsymbol{x}) + \frac{g}{2}(\phi^\dagger(\boldsymbol{x})\phi(\boldsymbol{x}))^2\right].$$

- Note that we renamed $\phi_{n=0}$ as ϕ.
- We have **decreased** the **dimension** of the system from $d+1$ to d.
- This is called a **dimensional reduction**. Therefore, we will work with $\boldsymbol{x}$ and $\boldsymbol{q}$, instead of $x=(\tau,\boldsymbol{x})$ and $q=(\omega_n,\boldsymbol{q})$.

J. Andersen, Rev. Mod. Phys. **76**, 599 (2004).

- This model is much **easier** to study, as there are **no Matsubara sums**.

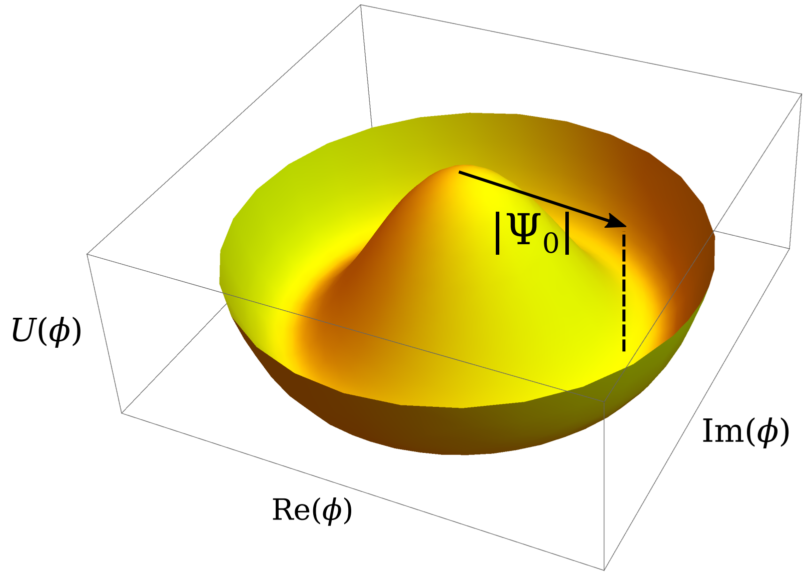
Real fields

- It is convenient to parametrise the fields as follows

$$\phi(\boldsymbol{x}) = \sqrt{MT}(\phi_1(\boldsymbol{x}) + i\,\phi_2(\boldsymbol{x})), \qquad \phi^\dagger(\boldsymbol{x}) = \sqrt{MT}(\phi_1(\boldsymbol{x}) - i\,\phi_2(\boldsymbol{x})).$$

- Therefore, we work with **real fields** ϕ_1 and ϕ_2 instead of the complex fields ϕ and $\phi^\dagger$.

- The **spontaneous symmetry breaking** is easy to implement with these real fields.



O(2)-model

- With the new fields, the **microscopic action** now reads

$$\mathcal{S}[\boldsymbol{\phi}] = \int d\boldsymbol{x} \left[\frac{1}{2} [\nabla \boldsymbol{\phi}(x)]^2 + \frac{r_0}{2} \boldsymbol{\phi}^2(x) + \frac{u_0}{4!} [\boldsymbol{\phi}^2(x)]^2 \right],$$

where $\boldsymbol{\phi}=(\phi_1,\phi_2)$, $\boldsymbol{\phi}^2=\phi_1^2+\phi_2^2$, $(\nabla\phi)^2=(\nabla\phi_1)^2+(\nabla\phi_2)^2$, and

$$r_0 = -2MT\mu, \qquad u_0 = 12M^2 gT.$$

- This is known as an **O(2)-model**. This can be done because the U(1)- and O(2)-symmetries are isomorphic.
- It enables us to study the **critical** behaviour of Bose gases.
- In the following, we will study this model with the **FRG**.

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Functional renormalisation group for the O(2)-model

- We aim to solve the **Wetterich equation**

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{tr}\left[\partial_k \mathcal{R}_k (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\right],$$

with the **microscopic action** as the **initial condition**.

- However, the Wetterich equation can rarely be solved exactly, as one generally has an infinite hierarchy of equations,

$$\Gamma_k[\varphi] = \sum_n \frac{1}{n!} \int_{x_1} \dots \int_{x_n} \Gamma_k^{(n)}[\varphi; x_1, \dots, x_n]\varphi(x_1)\dots\varphi(x_n).$$

$$\Gamma_k^{(n)} = \frac{\delta^n \Gamma_k}{\delta\varphi \dots \delta\varphi}$$

- In these cases, one proposes a **truncated ansatz** for Γ_k.
  - → Vertex expansion.
  - → **Derivative expansion**.

Classical fields
$\varphi = \langle \phi \rangle$

Derivative expansion

- In the **derivative expansion (DE)**, we **expand** the effective action in terms of the **fields** and **their derivatives**.

$$\varphi,\, \varphi^2,\, \varphi^4,\, ...,\, (\nabla\varphi)^2,\, (\nabla\varphi)^4,\, ...,\, \varphi^2(\nabla\varphi)^2, ...$$

- It works because the **integrals** are **dominated** by **small momenta** due to the effect of the regulator.
- The terms need to **respect the symmetries** of the original model.
- Despite the expansion, the **FRG is still non-perturbative**.
- This is because it is not an expansion in terms of the physical inputs, such as the interactions.

Local potential approximation

- The **lowest order** of the DE corresponds to the **local potential approximation (LPA)**.

- For the O(2)-model, the **LPA ansatz** reads

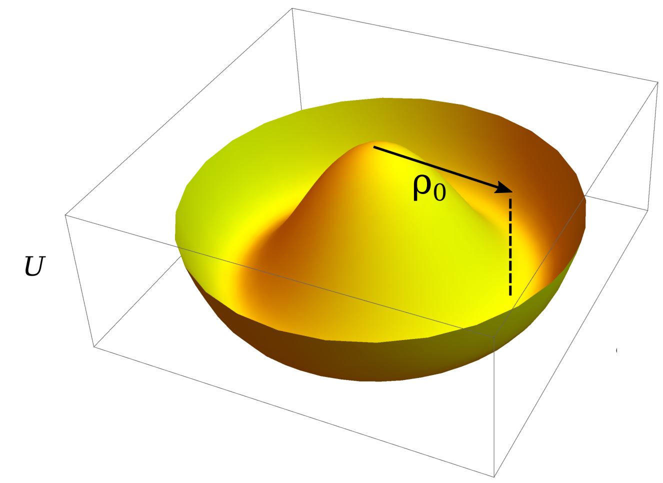
$$\Gamma_k[\boldsymbol{\varphi}] = \int d\boldsymbol{x} \left[\frac{1}{2} [\nabla \boldsymbol{\varphi}(x)]^2 + U_k(\rho(x)) \right],$$

Classical fields
$\varphi = \langle \phi \rangle$

where $\rho=(\varphi_1^2+\varphi_2^2)/2$ and

$$U_k(\rho) = u_{0,k} + m_k^2(\rho - \rho_{0,k}) + \frac{\lambda_k}{2}(\rho - \rho_{0,k})^2,$$

is the **effective potential**. This has been expanded around the **minimum** $\rho_{0,k}$.

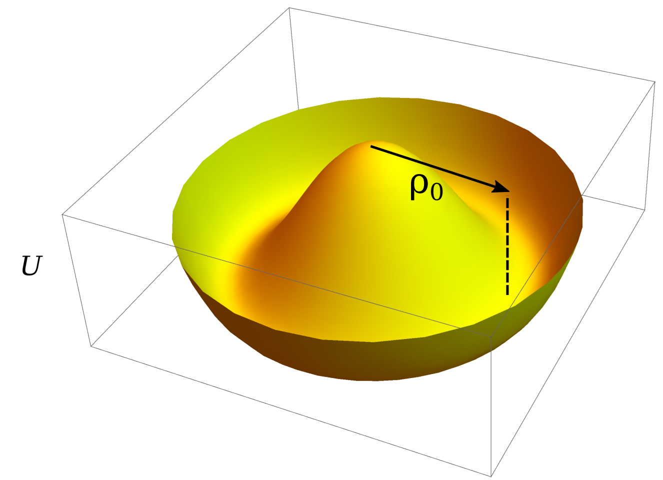


- The **couplings** $u_{k,0}$, $m^2{}_k$, λ_k, and $\rho_{0,k}$ **depend on the scale** k.

Effective potential

- The **equilibrium** condition reads

$$\left.\frac{\partial U_k}{\partial \rho}\right|_{\rho=\rho_{0,k}} = 0.$$



$\rho_0 = \varphi_0^2/2$ $\qquad \varphi(q) \longrightarrow \varphi_0\delta(q)$

Vacuum expectation value

- This means that

$$\rho_{0,k} > 0,\ m_k^2 = 0 \quad : \text{Symmetry is broken } (\textbf{broken phase})$$

$$\rho_{0,k} = 0,\ m_k^2 > 0 \quad : \text{Symmetry is not broken } (\textbf{symmetric phase})$$

- Therefore, the **physical phase of the system** is dictated by the values of $\rho_{0,k}$ and m_k^2 in the **physical limit** $k\to 0$.

- The **minimum** acts as the **order parameter**.

Flow equations

- We follow the flow of $u_{k,0}$, m_k^2, λ_k, and $\rho_{0,k}$, instead of that of Γ_k. Therefore, we need a **flow equation for each coupling**.
- These are obtained by differentiating the **Wetterich equation**.
- By noting that $\partial_k U_k = \partial_k \Gamma_k$, we have that

$$\partial_k U_k = \frac{1}{2} \mathrm{tr} \left[\partial_k \mathcal{R}_k (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \right] .$$

- Then, the **flow equations** are

$$\partial_k m_k^2 - \lambda_k \partial_k \rho_{0,k} = \left(\frac{\partial}{\partial \rho} \partial_k U_k \right) \Big|_{\rho=\rho_{0,k}} ,$$

$$\partial_k \lambda_k = \left(\frac{\partial^2}{\partial \rho^2} \partial_k U_k \right) \Big|_{\rho=\rho_{0,k}} .$$

Driving terms

- Note that when $\rho_{0,k}>0$, we set $m_k^2=0$, and viceversa.

Flow equations

- To compute the flow equation, we first write the **ansatz** in **momentum space**

$$
\begin{aligned}
\Gamma_k[\boldsymbol{\varphi}] = \int_{\boldsymbol{q}} \frac{\boldsymbol{q}^2}{2} & \left[\varphi_1(-\boldsymbol{q})\varphi_1(\boldsymbol{q}) + \varphi_2(-\boldsymbol{q})\varphi_2(\boldsymbol{q})\right] \\
& + m_k^2 \left(\frac{1}{2} \left[\int_{\boldsymbol{q}} \varphi_1(-\boldsymbol{q})\varphi_1(\boldsymbol{q}) + \varphi_2(-\boldsymbol{q})\varphi_2(\boldsymbol{q}) \right] - \rho_0 \right) \\
& + \frac{\lambda_k}{2} \left(\frac{1}{4} \left[\int_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{q}''} (\varphi_1(\boldsymbol{q})\varphi_1(\boldsymbol{q}')\varphi_1(\boldsymbol{q}'')\varphi_1(-\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'') \right.\right. \\
& \qquad + 2\varphi_1(\boldsymbol{q})\varphi_1(\boldsymbol{q}')\varphi_2(\boldsymbol{q}'')\varphi_2(-\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'') \\
& \qquad \left. + \varphi_2(\boldsymbol{q})\varphi_2(\boldsymbol{q}')\varphi_2(\boldsymbol{q}'')\varphi_2(-\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')) \right] \\
& \qquad \left. - \rho_0 \left[\int_{\boldsymbol{q}} \varphi_1(-\boldsymbol{q})\varphi_1(\boldsymbol{q}) + \varphi_2(-\boldsymbol{q})\varphi_2(\boldsymbol{q}) \right] - \rho_0 \right).
\end{aligned}
$$

Flow equations

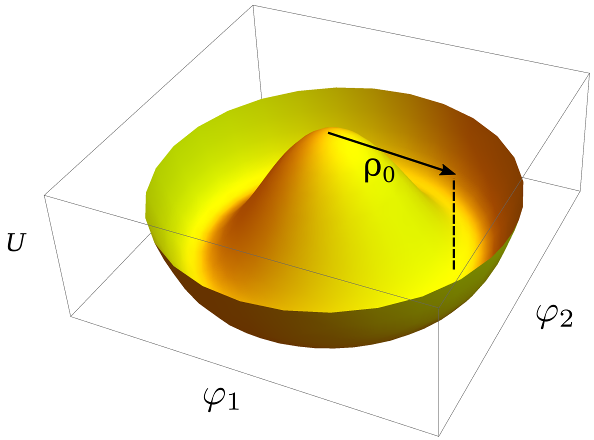
- The next step is to take **functional derivatives**,

$$\Gamma_k^{(2)}(\boldsymbol{q}) = \frac{\delta^2\Gamma_k}{\delta\boldsymbol{\varphi}(-\boldsymbol{q})\boldsymbol{\varphi}(\boldsymbol{q})} = \begin{pmatrix} \Gamma^{(2)}_{k,\varphi_1\varphi_1} & \Gamma^{(2)}_{k,\varphi_1\varphi_2} \\ \Gamma^{(2)}_{k,\varphi_2\varphi_1} & \Gamma^{(2)}_{k,\varphi_2\varphi_2} \end{pmatrix}.$$

$$\Gamma^{(2)}_{k,\varphi_i\varphi_i} = \frac{\delta^2\Gamma_k}{\delta\varphi_i(-\boldsymbol{q})\varphi_i(\boldsymbol{q})}$$

- After evaluating at the **background fields**

$$\varphi_1 = \sqrt{2\rho}, \qquad \varphi_2 = 0,$$



we obtain

$$\Gamma_k^{(2)}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{q}^2 + m_k^2 + \lambda_k(3\rho - \rho_{0,k}) & 0 \\ 0 & \boldsymbol{q}^2 + m_k^2 + \lambda_k(\rho - \rho_{0,k}) \end{pmatrix}.$$

Flow equations

- The **regulator** matrix is

$$\mathcal{R}_k(\boldsymbol{q}) = \begin{pmatrix} R_k(\boldsymbol{q}) & 0 \\ 0 & R_k(\boldsymbol{q}) \end{pmatrix}.$$

- Therefore, the **propagator** is

$$G_k(\boldsymbol{q}) = (\Gamma_k^{(2)}(\boldsymbol{q}) + \mathcal{R}_k(\boldsymbol{q}))^{-1} = \begin{pmatrix} \frac{1}{E_{1,k}(\boldsymbol{q};\rho)} & 0 \\ 0 & \frac{1}{E_{2,k}(\boldsymbol{q};\rho)} \end{pmatrix},$$

where

$$E_{1,k}(\boldsymbol{q};\rho) = \boldsymbol{q}^2 + m_k^2 + \lambda_k(3\rho - \rho_{0,k}) + R_k(\boldsymbol{q}),$$

$$E_{2,k}(\boldsymbol{q};\rho) = \boldsymbol{q}^2 + m_k^2 + \lambda_k(\rho - \rho_{0,k}) + R_k(\boldsymbol{q}),$$

are the **regulated energies**.

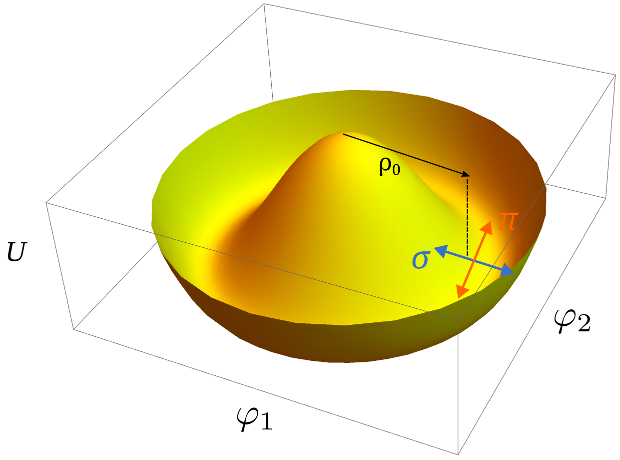
Flow equations

- Note that if we evaluate at the minimum $\rho = \rho_0$,

$$E_{1,k}(\boldsymbol{q};\rho_{0,k}) = \boldsymbol{q}^2 + m_k^2 + 2\lambda_k \rho_{0,k} + R_k(\boldsymbol{q}),$$

$$E_{2,k}(\boldsymbol{q};\rho_{0,k}) = \boldsymbol{q}^2 + m_k^2 + R_k(\boldsymbol{q}).$$

- Therefore, the **real direction** φ_1 represents the longitudinal **Higgs mode**.
- In contrast, the **imaginary direction** φ_2 represents the gapless **Goldstone mode**.



Flow equations (driving terms)

- Finally, we have that

$$\partial_k U_k = \frac{1}{2}\mathrm{tr}\left[\partial_k \mathcal{R}_k(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\right] = \frac{1}{2}\int_{\boldsymbol{q}} \left(\underbrace{\frac{1}{E_{1,k}(\boldsymbol{q};\rho)}}_{\text{Higgs}} + \underbrace{\frac{1}{E_{2,k}(\boldsymbol{q};\rho)}}_{\text{Goldstone}} \right) \partial_k R_k(\boldsymbol{q}).$$

- By differentiating, we obtain the **driving terms**

$$\left(\frac{\partial}{\partial\rho}\partial_k U_k\right)\Big|_{\rho=\rho_{0,k}} = -\frac{\lambda_k}{2}\int_{\boldsymbol{q}} \left(\frac{3}{E_{1,k}^2(\boldsymbol{q};\rho_{0,k})} + \frac{1}{E_{2,k}^2(\boldsymbol{q};\rho_{0,k})} \right) \partial_k R_k(\boldsymbol{q}),$$

$$\left(\frac{\partial^2}{\partial\rho^2}\partial_k U_k\right)\Big|_{\rho=\rho_{0,k}} = \frac{\lambda_k^2}{2}\int_{\boldsymbol{q}} \left(\frac{9}{E_{1,k}^3(\boldsymbol{q};\rho_{0,k})} + \frac{1}{E_{2,k}^2(\boldsymbol{q};\rho_{0,k})} \right) \partial_k R_k(\boldsymbol{q}).$$

- In principle, these are **integro-differential equations**.
- To solve the flow, we need **initial conditions** and a **regulator**.

Initial conditions

- For the initial conditions, we set

$$\Gamma_\Lambda[\varphi] = \mathcal{S}[\varphi].$$

- Then,

$$\frac{1}{2}[\nabla\boldsymbol{\varphi}(x)]^2 + U_\Lambda(\rho(x)) = \frac{1}{2}[\nabla\boldsymbol{\varphi}(x)]^2 + \frac{r_0}{2}\boldsymbol{\varphi}^2(x) + \frac{u_0}{4!}[\boldsymbol{\varphi}^2(x)]^2.$$

- For the relevant case of positive chemical potential (r_0=-$2mT\mu$<0), we obtain the following **initial conditions**

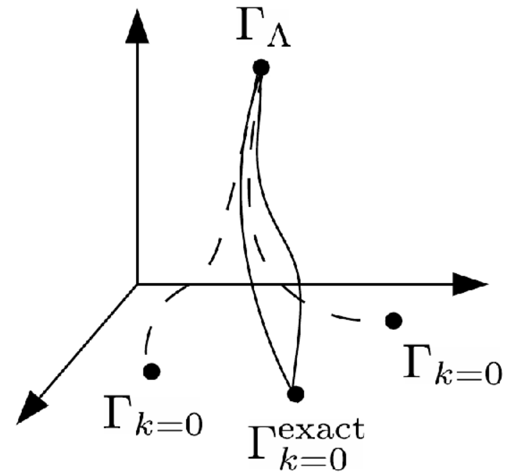
$$m_\Lambda^2 = 0, \qquad \rho_{0,\Lambda} = -3r_0/u_0, \qquad \lambda_\Lambda = u_0/3.$$

- Therefore, the **flow starts** in the **broken phase**.

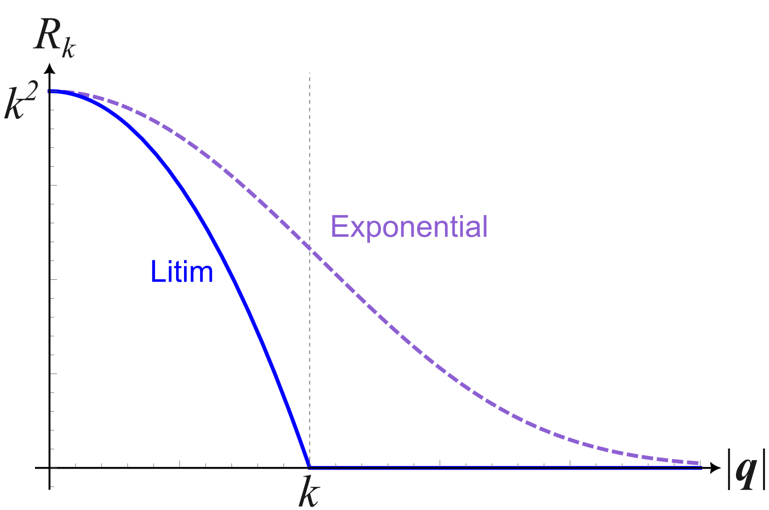
Regulator

- In principle, the physical results should not depend on the regulator choice.
- However, due to the truncation of the ansatz, the **results** do **depend on the regulator**.
  J. Pawloski *et al.*, Ann. Phys (N.Y.) **384**, 165 (2017).
- There are different typical choices. One is the **exponential regulator**

$$R_k(\boldsymbol{q}) = \frac{\boldsymbol{q}^2}{\exp(\boldsymbol{q}^2/k^2) - 1}.$$



Different regulators results in different solutions. Figure taken from N. Dupuis *et al.*, Phys. Rep. **910**, 1 (2021).



- In these lectures, we will use the **optimised Litim regulator**
  D. Litim, Phys. Lett. B **486**, 92 (2000).

$$R_k(\boldsymbol{q}) = (k^2 - \boldsymbol{q}^2)\Theta(k^2 - \boldsymbol{q}^2).$$

Θ : Heaviside step function

Flow equations (driving terms)

- For the employed ansatz, the **Litim regulator** enables us to **integrate** the momentum integrals **analytically**.
  - → Differential equations instead of integro-differential ones.
- The k-derivative of the Litim regulator

$$\partial_k R_k(\boldsymbol{q}) = 2k\Theta(k^2 - \boldsymbol{q}^2).$$

- Therefore, the integrals are confined to $|\boldsymbol{q}|\in[0,k]$, where the regulated energies become independent of $\boldsymbol{q}$,

$$R_k(\boldsymbol{q}) = (k^2 - \boldsymbol{q}^2)\Theta(k^2 - \boldsymbol{q}^2).$$

$$E_{1,k}(\boldsymbol{q};\rho_{0,k}) = \boldsymbol{q}^2 + m_k^2 + 2\lambda_k\rho_{0,k} + R_k(\boldsymbol{q}) = \boxed{k^2 + m_k^2 + 2\lambda_k\rho_{0,k},}$$

$E_{1,k}(k;\rho_{0,k})$

$$E_{2,k}(\boldsymbol{q};\rho_{0,k}) = \boldsymbol{q}^2 + m_k^2 + R_k(\boldsymbol{q}) = \boxed{k^2 + m_k^2.}$$

$E_{2,k}(k)$

Flow equations (driving terms)

- Therefore, for the **Litim regulator**, the **driving terms** read

$$\left(\frac{\partial}{\partial\rho}\partial_k U_k\right)\bigg|_{\rho=\rho_{0,k}} = -\frac{\lambda_k}{2}\left(\frac{3}{E^2_{1,k}(k;\rho_{0,k})}+\frac{1}{E^2_{2,k}(k)}\right)2k\int_{|\boldsymbol{q}|<k},$$

$$\left(\frac{\partial^2}{\partial\rho^2}\partial_k U_k\right)\bigg|_{\rho=\rho_{0,k}} = \frac{\lambda_k^2}{2}\left(\frac{9}{E^3_{1,k}(k;\rho_{0,k})}+\frac{1}{E^2_{2,k}(k)}\right)2k\int_{|\boldsymbol{q}|<k}.$$

- The remaining integral is

$$\int_{|\boldsymbol{q}|<k} = \frac{S_d}{(2\pi)^d}\int_0^k q^{d-1}dq = \frac{S_d}{(2\pi)^d}\frac{k^d}{d},$$

where $S_d{=}2\pi^{d/2}/\Gamma(d/2)$. Thus,

$$\int_{|\boldsymbol{q}|<k} \overset{d\,=\,3}{=} \frac{1}{(2\pi)^3}\frac{4\pi}{3}k^3, \qquad \int_{|\boldsymbol{q}|<k} \overset{d\,=\,2}{=} \frac{1}{(2\pi)^2}\pi k^2.$$

*The volume has been absorbed into the effective action.

Flow equations

- Finally, the full **flow equations** for the **Litim regulator** are

$$\partial_k m_k^2 - \lambda_k \partial_k \rho_{0,k} = -\lambda_k \left(\frac{3}{(k^2 + m_k^2 + 2\lambda_k \rho_{0,k})^2} + \frac{1}{(k^2 + m_k^2)^2} \right) \frac{S_d}{(2\pi)^d} \frac{k^{d+1}}{d},$$

$$\partial_k \lambda_k = \lambda_k^2 \left(\frac{9}{(k^2 + m_k^2 + 2\lambda_k \rho_{0,k})^3} + \frac{1}{(k^2 + m_k^2)^3} \right) \frac{S_d}{(2\pi)^d} \frac{k^{d+1}}{d}.$$

- These can be **solved numerically** with standard routines for numerical integration.
- Note that we start with $\rho_{0,k} > 0$, and fix $m_k^2 = 0$. If $\rho_{0,k}$ vanishes at a finite scale k_s, then m_k^2 is allowed to flow for $k < k_s$.

Flow equations (two last remarks)

- In general, one solves the flow as a function of the **RG time**

$$t = \log(k/\Lambda),$$

  from $t_\Lambda = 0$ to $t \to \infty$.

- Additionally, the flow has to start at a **high UV scale above the relevant scales** of the system.

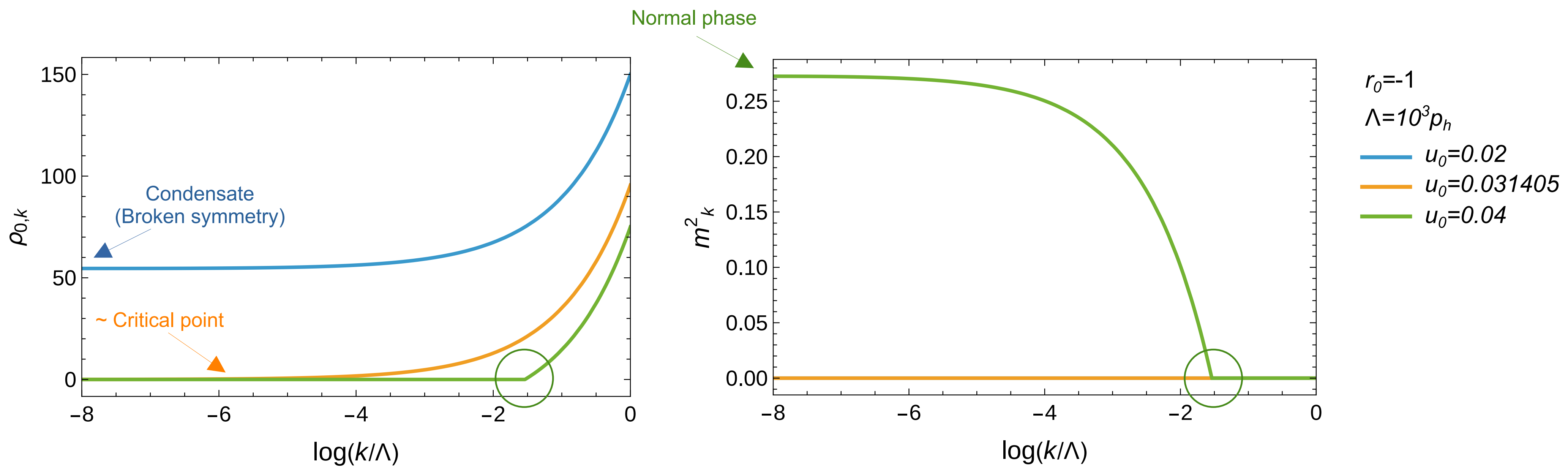
- In our model, the only scale is given by the

$$p_h = \sqrt{2|r_0|} \qquad \longrightarrow \qquad \Lambda \gg p_h .$$

- This ensures that at high scales

$$E_{1,k}(k;\rho_{0,k}) = k^2 + 2\lambda_k \rho_{0,k} \approx k^2 .$$

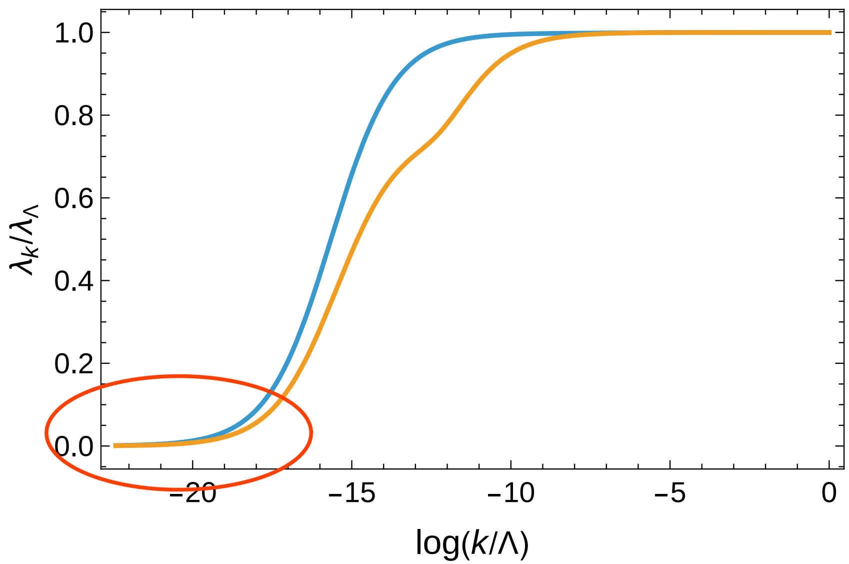
Flow examples (d=3)



- The blue lines show a flow for a **symmetry-broken system**, while the green lines show a flow for a **symmetric system**.
- The **critical point** corresponds to the flow where ρ_0 vanishes exactly at $k=0$. A flow near the **phase transition** is shown by the orange lines.

When to stop the flow

- The **couplings** should **saturate** to a **finite value** or to their **asymptotic behaviour** for small k.



- This **saturation** is reached **below the relevant scales** of the flow.

$$E_{1,k}(k;\rho_{0,k}) = k^2 + m_k^2 + 2\lambda_k\rho_{0,k} \quad \longrightarrow \quad k^2_{\text{finish}} \ll \boxed{m_k^2 + 2\lambda_k\rho_{0,k}.}$$

Lecture 2

1. Bose gases and the **O(2)-model**.
2. The **FRG for the O(2)-model**. LPA and the **derivative expansion**.
3. **Beyond LPA** and **critical** behaviour.
4. Summary.

LPA'

- The first improvement to the **LPA** is to consider a flowing **field renormalisation** Z_k.
- This is referred to as **LPA'**. The new **ansatz** reads

$$\Gamma_k[\varphi] = \int d\boldsymbol{x} \left[\frac{Z_k}{2} [\nabla \boldsymbol{\varphi}(x)]^2 + U_k(\rho(x)) \right],$$

  where the effective potential remains unchanged.
- The coupling Z_k flows with k, and thus it has its own **flow equation**.
- Its initial condition is

$$Z_\Lambda = 1.$$

LPA' (regulated energies and regulator)

- The propagator maintains its form with modified **regulated energies**

$$E_{1,k}(\boldsymbol{q};\rho) = Z_k\boldsymbol{q}^2 + m_k^2 + \lambda_k(3\rho - \rho_{0,k}) + R_k(\boldsymbol{q}),$$

$$E_{2,k}(\boldsymbol{q};\rho) = Z_k\boldsymbol{q}^2 + m_k^2 + \lambda_k(\rho - \rho_{0,k}) + R_k(\boldsymbol{q}).$$

- The **regulator** is also changed. The **Litim regulator** now reads

$$R_k(\boldsymbol{q}) = Z_k(k^2 - \boldsymbol{q}^2)\Theta(k^2 - \boldsymbol{q}^2).$$

LPA' (driving terms)

- Firstly, we revise the **flow equations** for the **effective potential**.
- The **driving terms** have the same form

$$\left(\frac{\partial}{\partial\rho}\partial_k U_k\right)\Big|_{\rho=\rho_{0,k}} = -\frac{\lambda_k}{2}\int_{\boldsymbol{q}}\left(\frac{3}{E_{1,k}^2(\boldsymbol{q};\rho_{0,k})}+\frac{1}{E_{2,k}^2(\boldsymbol{q})}\right)\partial_k R_k(\boldsymbol{q}),$$

$$\left(\frac{\partial^2}{\partial\rho^2}\partial_k U_k\right)\Big|_{\rho=\rho_{0,k}} = \frac{\lambda_k^2}{2}\int_{\boldsymbol{q}}\left(\frac{9}{E_{1,k}^3(\boldsymbol{q};\rho_{0,k})}+\frac{1}{E_{2,k}^2(\boldsymbol{q})}\right)\partial_k R_k(\boldsymbol{q}).$$

- However, in addition to the changes of the regulated energies, the derivative of the regulator also contains a term with $\partial_k Z_k$.
- For the **Litim regulator**

$$\partial_k R_k(\boldsymbol{q}) = 2kZ_k\Theta(k^2-\boldsymbol{q}^2) + (k^2-\boldsymbol{q}^2)\Theta(k^2-\boldsymbol{q}^2)\partial_k Z_k.$$

LPA' (driving terms)

- The new **driving terms** read

$$\left(\frac{\partial}{\partial\rho}\partial_k U_k\right)\bigg|_{\rho=\rho_{0,k}} = -\frac{\lambda_k}{2}\left(\frac{3}{E_{1,k}^2(k;\rho_{0,k})}+\frac{1}{E_{2,k}^2(k)}\right)\left[(2kZ_k+k^2\partial_k Z_k)\int_{|\boldsymbol{q}|<k} - \partial_k Z_k\int_{|\boldsymbol{q}|<k}\boldsymbol{q}^2\right],$$

$$\left(\frac{\partial^2}{\partial\rho^2}\partial_k U_k\right)\bigg|_{\rho=\rho_{0,k}} = \frac{\lambda_k^2}{2}\left(\frac{9}{E_{1,k}^3(k;\rho_{0,k})}+\frac{1}{E_{2,k}^2(k)}\right)\left[(2kZ_k+k^2\partial_k Z_k)\int_{|\boldsymbol{q}|<k} - \partial_k Z_k\int_{|\boldsymbol{q}|<k}\boldsymbol{q}^2\right].$$

- One obtains

$$\partial_k m_k^2 - \lambda_k\partial_k\rho_{0,k} = -\lambda_k\left(\frac{3}{(Z_k k^2+m_k^2+2\lambda_k\rho_{0,k})^2}+\frac{1}{(Z_k k^2+m_k^2)^2}\right)\frac{S_d}{(2\pi)^d}\frac{k^{d+1}}{d}\left(Z_k+\frac{k}{d+2}\partial_k Z_k\right),$$

$$\partial_k\lambda_k = \lambda_k^2\left(\frac{9}{(Z_k k^2+m_k^2+2\lambda_k\rho_{0,k})^3}+\frac{1}{(Z_k k^2+m_k^2)^3}\right)\frac{S_d}{(2\pi)^d}\frac{k^{d+1}}{d}\left(Z_k+\frac{k}{d+2}\partial_k Z_k\right).$$

LPA' (driving terms)

- Now, we compute the **driving terms** for Z_k. It is obtained from

$$\partial_k Z_k = \left(\frac{\partial}{\partial \boldsymbol{p}^2} \partial_k \Gamma^{(2)}_{k,\varphi_2\varphi_2}(\boldsymbol{p}) \right) \Bigg|_{\rho=\rho_{0,k},\boldsymbol{p}=0},$$

Driving term

where

$$\Gamma^{(2)}_{k,\varphi_i\varphi_i}(\boldsymbol{p}) = \frac{\delta^2 \Gamma_k}{\delta\varphi_i(-\boldsymbol{p})\varphi_i(\boldsymbol{p})}.$$

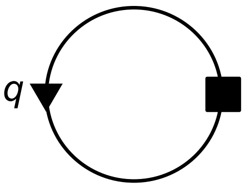
- Therefore, we need the flow of the **two-point function** $\Gamma_k^{(2)}$.
- Also note that we extract the flow from the **Goldstone sector** (φ_2,φ_2).
- The reason for the latter will become clear later in these lectures.

One-point function

- We start from the **Wetterich equation**

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{tr} \left[\partial_k \mathcal{R}_k(\boldsymbol{q}) G_k(\boldsymbol{q}) \right].$$

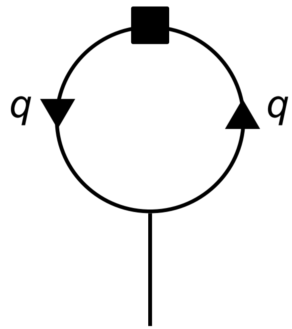
$$G_k(\boldsymbol{q}) = (\Gamma_k^{(2)}(\boldsymbol{q}) + \mathcal{R}_k(\boldsymbol{q}))^{-1}$$



q: internal momentum

- By taking one functional derivative, the flow of the **one-point function** $\Gamma_k^{(1)}$ is dictated by

$$\begin{aligned} \partial_k \Gamma^{(1)}_{k,\varphi_i} &= \frac{\delta}{\delta \varphi_i(0)} \partial_k \Gamma_k \\ &= -\frac{1}{2} \mathrm{tr} \left[\partial_k \mathcal{R}_k(\boldsymbol{q}) G_k(\boldsymbol{q}) \Gamma^{(3)}_{k,\varphi_i}(0, \boldsymbol{q}, -\boldsymbol{q}) G_k(\boldsymbol{q}) \right], \end{aligned}$$



q: internal momentum

where we have used that

$$\delta_{\varphi_i} G_k = -G_k \Gamma_k^{(3)} G_k, \qquad \Gamma^{(3)}_{k,\varphi_i} = \frac{\delta \Gamma_k^{(2)}}{\delta \varphi_i}.$$

Two-point function

- The flow equation of the **two-point function** $\Gamma_k^{(1)}$ is obtained by taking a second functional derivative

$$\partial_k \Gamma^{(2)}_{k,\varphi_i\varphi_j}(\boldsymbol{p}) = \frac{\delta^2}{\delta\varphi_j(-\boldsymbol{p})\delta\varphi_i(\boldsymbol{p})}\partial_k\Gamma_k$$

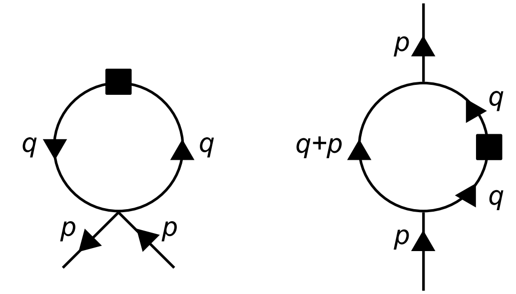
$$= -\frac{1}{2}\mathrm{tr}\left[\partial_k\mathcal{R}_k(\boldsymbol{q})G_k(\boldsymbol{q})\Gamma^{(4)}_{k,\varphi_i\varphi_j}(\boldsymbol{p},-\boldsymbol{p},\boldsymbol{q},-\boldsymbol{q})G_k(\boldsymbol{q})\right]$$

$$+\mathrm{tr}\left[\partial_k\mathcal{R}_k(\boldsymbol{q})G_k(\boldsymbol{q})\Gamma^{(3)}_{k,\varphi_i}(\boldsymbol{p},\boldsymbol{q},-\boldsymbol{q}-\boldsymbol{p})G_k(\boldsymbol{q}+\boldsymbol{p})\Gamma^{(3)}_{k,\varphi_j}(-\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p},-\boldsymbol{q})G_k(\boldsymbol{q})\right],$$

where

$$\Gamma^{(4)}_{k,\varphi_i\varphi_j} = \frac{\delta\Gamma^{(2)}_k}{\delta\varphi_j\delta\varphi_i}.$$

- In **diagrammatic form**



q: internal momentum

p: external momentum

Vertices

- The relevant **vertices** are

$$\Gamma^{(3)}_{k,\phi_2} = \begin{pmatrix} 0 & 2\sqrt{\lambda_k \rho} \\ 2\sqrt{\lambda_k \rho} & 0 \end{pmatrix},$$

  and

$$\Gamma^{(4)}_{k,\phi_2\phi_2} = \begin{pmatrix} \lambda_k & 0 \\ 0 & 3_k \end{pmatrix}.$$

- Both are **momentum-independent**.
- Therefore, for this ansatz, we do not have to worry about the correct order of the momenta.
- Note that the **evaluation** at the **background fields** must be done **after** taking all the **functional derivatives**.

Two-point function

- After some algebra, one obtains

$$\partial_k \Gamma^{(2)}_{k,\varphi_2\varphi_2}(\boldsymbol{p})\Big|_{\rho=\rho_{0,k}} = 2\rho_0\lambda_k \int_{\boldsymbol{q}} \left(\frac{1}{E_{1,k}(\boldsymbol{q}+\boldsymbol{p};\rho_{0,k})E^2_{2,k}(\boldsymbol{q})} + \frac{1}{E^2_{1,k}(\boldsymbol{q};\rho_{0,k})E_{2,k}(\boldsymbol{q}+\boldsymbol{p})} \right) \partial_k R_k(\boldsymbol{q}).$$

- Now, we need to take the $\boldsymbol{p}^2$-derivative.

$$\partial_k Z_k = \left(\frac{\partial}{\partial \boldsymbol{p}^2} \partial_k \Gamma^{(2)}_{k,\varphi_2\varphi_2}(\boldsymbol{p}) \right) \Big|_{\rho=\rho_{0,k},\boldsymbol{p}=0}.$$

- By writing the i**nternal and external momenta** as

$$\boldsymbol{q} = (q,0,0,...), \qquad \boldsymbol{p} = (p_x,p_y,p_z,...),$$

the derivative takes the form

$$\partial_k Z_k = \frac{1}{2d}\sum_{i=1}^{d} \left(\frac{\partial^2}{\partial p_i^2} \partial_k \Gamma^{(2)}_{k,\varphi_2\varphi_2}(\boldsymbol{p}) \right) \Big|_{\rho=\rho_{0,k},\boldsymbol{p}=0}.$$

LPA' (driving terms)

- One obtains

$$\partial_k Z_k = -\frac{2\rho_{0,k}\lambda_k^2}{d}\int_{\boldsymbol{q}} \frac{1}{E_{1,k}^3(\boldsymbol{q};\rho_0)E_{2,k}^3(\boldsymbol{q})}\partial_k R_k(\boldsymbol{q})\Big(-4\boldsymbol{q}^2\left(E_{2,k}(\boldsymbol{q})(E_{1,k}^{\mathrm{x}}(\boldsymbol{q}))^2 + E_{1,k}(\boldsymbol{q};\rho_0)(E_{2,k}^{\mathrm{x}}(\boldsymbol{q}))^2\right)$$

$$+E_{1,k}(\boldsymbol{q};\rho_0)E_{2,k}(\boldsymbol{q}^2)\left(d(E_{1,k}^{\mathrm{x}}(\boldsymbol{q}) + E_{2,k}^{\mathrm{x}}(\boldsymbol{q})) + 2q^2(E_{1,k}^{\mathrm{xx}}(\boldsymbol{q}) + E_{2,k}^{\mathrm{xx}}(\boldsymbol{q}))\right),$$

where one x superscript denotes one $\boldsymbol{q}^2$-derivative. Thus,

$$E_{i,k}^{\mathrm{x}}(\boldsymbol{q}) = \frac{\partial}{\partial \boldsymbol{q}^2}E_{i,k}(\boldsymbol{q}) = Z_k + R_k^{\mathrm{x}}(\boldsymbol{q}), \qquad E_{i,k}^{\mathrm{xx}}(\boldsymbol{q}) = \frac{\partial^2}{\partial(\boldsymbol{q}^2)^2}E_{i,k}(\boldsymbol{q}) = R_k^{\mathrm{xx}}(\boldsymbol{q}).$$

- For the **Litim regulator**, we note that

$$R_k^{\mathrm{x}}(\boldsymbol{q}) = -Z_k\Theta(k^2-\boldsymbol{q}^2), \qquad R_k^{\mathrm{xx}}(\boldsymbol{q}) = \frac{Z_k}{k^2}\delta(k^2-\boldsymbol{q}^2).$$

- The previous expressions simplify to

$$E_{i,k}^{\mathrm{x}}(\boldsymbol{q}) = 0, \qquad E_{i,k}^{\mathrm{xx}}(\boldsymbol{q}) = \frac{Z_k}{k^2}\delta(k^2-\boldsymbol{q}^2),$$

LPA' (driving terms)

- Finally, we get the simpler expression

$$\partial_k Z_k = -\frac{8\rho_{0,k}\lambda_k^2}{d}\int_{\boldsymbol{q}} \frac{\boldsymbol{q}^2 R_k^{\mathrm{xx}}}{E_{1,k}^2(\boldsymbol{q}^2;\rho_{0,k})E_{2,k}^2(\boldsymbol{q})}\partial_k R_k(\boldsymbol{q}).$$

- After integrating, we finally obtain

$$\partial_k Z_k = -\frac{8\rho_{0,k}\lambda_k^2}{d}\frac{S_d}{(2\pi)^d}\frac{Z_k^2 k^d}{(Z_k^2 + m_k^2 + 2\lambda_k\rho_{0,k})^2)(Z_k k^2 + m_k^2)^2}.$$

- With this, we now have analytical expressions for all the **flow equations**.

Anomalous dimension

- We can extract the **anomalous dimension** η from the renormalisation factor as

$$\eta_k = -k\,\partial_k \log(Z_k).$$

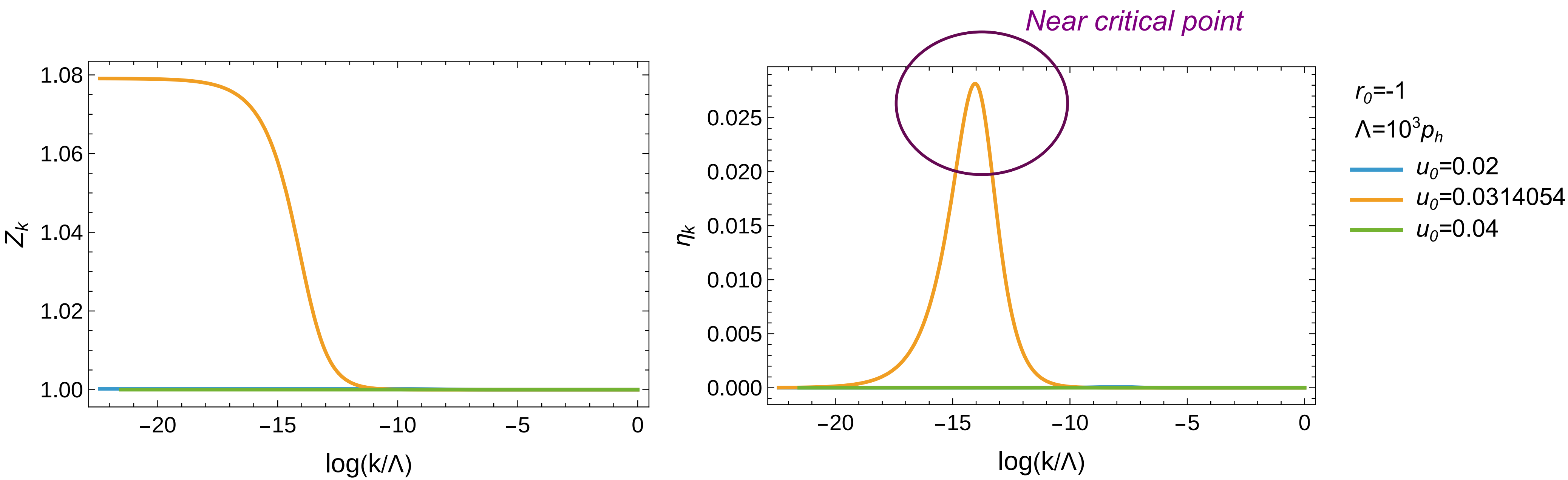
- At the **critical point**, the **correlation function** behaves as

$$G(\boldsymbol{p}) \propto \frac{1}{|\boldsymbol{p}|^{2-\eta}}. \qquad \eta > 0$$

- Therefore, η is a **critical exponent**. Then, at small k, the renormalisation factor should diverge as

$$Z_k \sim k^{-\eta}.$$

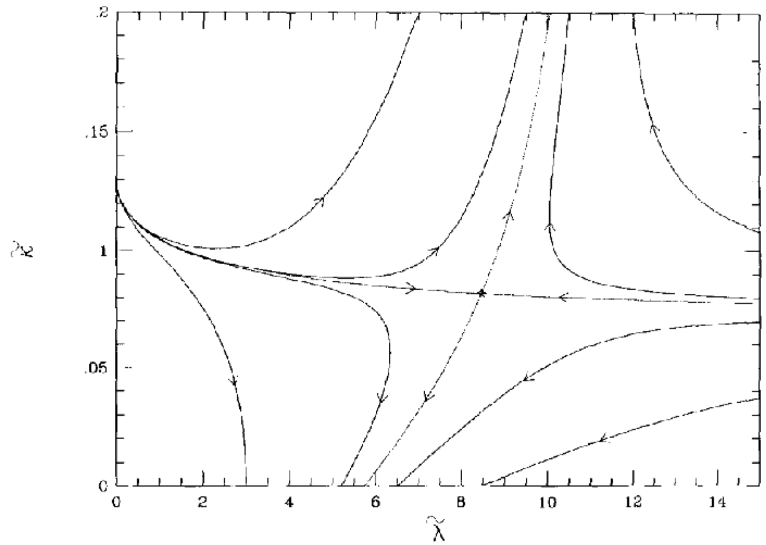
Flow examples (d=3)



- We can extract the **critical anomalous dimension** by examining the **peaks** of η very near the phase transition.
- QMC calculations report a value of $\eta \approx 0.036$.
  M. Hasenbusch, Phys. Rev. B **82**, 174433 (2010).
- Therefore, this simple FRG calculation gives a good estimate.

Common strategy

- It is common to **rescale the couplings** by their known scaling (asymptotic) behaviour.
- It is also common to follow the flow of η, instead of that of Z_k.
- This is related to the concept of **fixed points**.



Flows of rescaled ρ_0 and λ for an O(4)-model. Figure take from N. Tetradis and C. Wetterich, Nucl. Phys. B 398, 659 (1993).

- The **critical exponents** can be found **self-consistently** within a chosen threshold.

Dimensionality

- The **DE works** very **well** in **three dimensions**.
- **Fluctuations** become more **important** in **lower dimensions**, but the DE still (somewhat) works in **two dimensions**.
- The **Berezinskii-Kosterlitz-Thouless** (BKT) transition can be studied through a line of pseudo-fixed points.

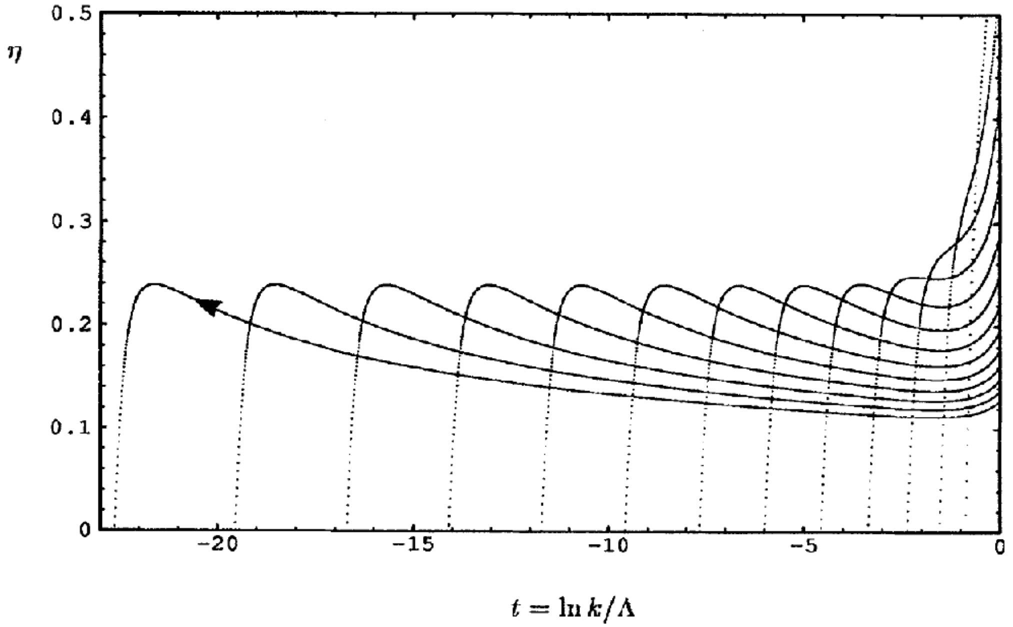


Illustration of the flow of near the BKT transition. Figure taken from M. Gräter and C. Wetterich, Phys. Rev. Lett. **75**, 378 (1995).

Next orders of the DE

- **Beyond LPA'**, the next step is to consider

$$\Gamma_k[\boldsymbol{\varphi}] = \int d\boldsymbol{x} \left[\frac{Z_k}{2} [\nabla \boldsymbol{\varphi}(x)]^2 + \frac{Y_k}{4} [\nabla \rho(x)]^2 + U_k(\rho(x)) \right].$$

- The initial condition for the new coupling is $Y_\Lambda = 0$.
- Unfortunately, for this ansatz, the integrals cannot be solved analytically with the **Litim regulator**.
- Therefore, we need to solve the full **integro-differential equations**.

Next orders of the DE

- Within this new truncation, the two-point function reads

$$\Gamma_k^{(2)}(\boldsymbol{q}) = \begin{pmatrix} (Z_k + Y_k\rho_0)\boldsymbol{q}^2 + m_k^2 + 2\lambda_k\rho_{0,k}) & 0 \\ 0 & Z_k\boldsymbol{q}^2 + m_k^2 \end{pmatrix}.$$

  - $Z_{\sigma,k} = Z_k + Y_k\rho_{0,k}$ acts as the renormalisation of the **Higgs modes**.
  - $Z_{\pi,k} = Z_k$ acts as the renormalisation of the **Goldstone modes**.

- This explains why the flow of Z_k needs to be extracted from the Goldstone part of the two-point function.

Next orders of the DE

- Further truncations include higher-order couplings

$$\Gamma_k[\boldsymbol{\varphi}] = \int d\boldsymbol{x} \left[\frac{Z_k(\rho(x))}{2} [\nabla \boldsymbol{\varphi}(x)]^2 + \frac{Y_k(\rho(x))}{4} [\nabla \rho(x)]^2 + U_k(\rho(x)) \right],$$

where

$$U_k(\rho) = \sum_{n=0}^{n_{\max,U}} \frac{u_{n,k}}{n!} (\rho - \rho_{0,k})^n,$$

$$Z_k(\rho) = \sum_{n=0}^{n_{\max,Z}} \frac{Z_{n,k}}{n!} (\rho - \rho_{0,k})^n,$$

$$Y_k(\rho) = \sum_{n=0}^{n_{\max,Y}} \frac{Y_{n,k}}{n!} (\rho - \rho_{0,k})^n.$$

- At **fourth order**, one can obtain **accurate results** for the critical exponents.

G. De Polsi, I. Balog, M. Tissier, and N. Wschebor, Phys. Rev. E **101**, 042113 (2020).

Lecture 2

1. Bose gases and the **O(2)-model**.
2. The **FRG for the O(2)-model**. LPA and the **derivative expansion**.
3. **Beyond LPA** and **critical** behaviour.
4. Summary.

Summary

- We have examined the use of the **FRG** for the classical **O(2)-model**.
- We have revised the general **FRG** strategy to compute the **flow equations** and solve the RG flow.
- Even a simple ansatz within the **derivative expansion** can give a good qualitative **description** of the **phase transition**.
- Next lecture:
  - → Bose and Fermi gases.