Auxiliar 18

P1] La amplitud de un oscilador amortiguado decrece en n periodos a $\frac{1}{e}$ de su valor inicial. Muestre que su frecuencia es $\left[1+\frac{1}{4\pi^2 n^2}\right]^{-1/2}$ veces la del oscilador sin roce.

Sol: Se tiene,

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \omega_0^2 x = 0$$

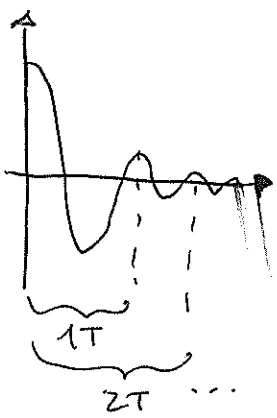
$$\rightarrow \lambda^2 + \frac{\gamma}{m}\lambda + \omega_0^2 = 0$$

$$\rightarrow \lambda = \frac{-\frac{\gamma}{m} \pm \sqrt{\frac{\gamma^2}{m^2} - 4\omega_0^2}}{2} = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2}$$

En el caso amortiguado: $\omega_0^2 > \left(\frac{\gamma}{2m}\right)^2$

$$\Rightarrow x = \left[Ae^{i\omega t} + Be^{-i\omega t}\right]e^{-\frac{\gamma}{2m}t} \qquad \text{donde } \omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

$$= \left[C\operatorname{sen}(\omega t) + D\cos(\omega t)\right]e^{-\frac{\gamma}{2m}t} \quad \rightarrow x(0) = D$$



En n periodos:

$$x_n = \frac{D}{e} = \left[C\operatorname{sen}(\omega nT) + D\cos(\omega nT)\right]e^{-\frac{\gamma}{2m}nT}$$

pero $T = \frac{2\pi}{\omega}$:

$$\frac{D}{e} = \left[\cancel{C\operatorname{sen}(2\pi n)} + D\underbrace{\cos(2\pi n)}_{1}\right]e^{-\frac{\gamma}{2m}\frac{2\pi n}{\omega}}$$

$$\cancel{D}e^{-1} = \cancel{D}e^{-\frac{\gamma\pi n}{\omega m}}$$

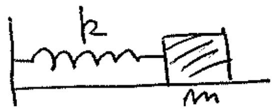
$$\Rightarrow 1 = \frac{\gamma\pi n}{\omega m} \Rightarrow \gamma = \frac{\omega m}{\pi n}$$

usando: $\omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$

$$\omega^2 = \omega_0^2 - \frac{\omega^2}{(2\pi n)^2} \longrightarrow \omega^2\left[1 + \frac{1}{4\pi^2 n^2}\right] = \omega_0^2$$

$$\Rightarrow \boxed{\frac{\omega}{\omega_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{-1/2}}$$

P2



$F_v = -bv$

$F_d = F_d \cos(\omega_d t)$

Encontrar $x(t)$

Sol: La ec. de movimiento:

$$m\ddot{x} = -b\dot{x} - kx + F_d\cos(\omega_d t)$$

$$\rightarrow \quad \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F\cos(\omega_d t)$$

$$= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)$$

donde $2\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$, $F = \frac{F_d}{m}$

Decimos que $x_p = A\cos(\omega_d t)$:

$$\Rightarrow \quad -A\omega_d^2\cos(\omega_d t) - 2\gamma\omega_d A\,\mathrm{sen}(\omega_d t) + \omega_0^2 A\cos(\omega_d t) = F\cos(\omega_d t)$$

$$\cos(\omega_d t)\left[\omega_0^2 A - \omega_d^2 A - F\right] = 2\gamma\omega_d A\,\mathrm{sen}(\omega_d t)$$

en $t=0$: $\quad A = \dfrac{F}{\omega_0^2 - \omega_d^2}$

$$\Rightarrow \quad x_p = \frac{F}{\omega_0^2 - \omega_d^2}\cos(\omega_d t)$$

La solución homogénea es conocida:

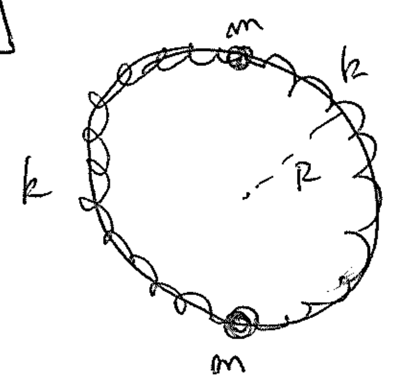
$$x_h = \left[C\,\mathrm{sen}(\omega^* t) + D\cos(\omega^* t)\right]e^{-\frac{\gamma}{m}t}$$

con $\omega^* = \sqrt{\omega_0^2 - \frac{\gamma^2}{m^2}}$

y la sol. general:

$$x = x_h + x_p$$

P3



Sol: Teníamos (de la Aux (16):

(1) $\ddot{\phi}_1 = 2\omega_0^2(-\phi_1+\theta_2)$

(2) $\ddot{\theta}_2 = 2\omega_0^2(\phi_1-\theta_2)$ con $\theta_2 = \phi_2 - \pi$

Sumamos (1)+(2):

$$\ddot{\phi}_1 + \ddot{\theta}_2 = 0$$

llamamos: $x_1 = \phi_1 + \theta_2 \Rightarrow \ddot{x}_1 = 0$

$\Rightarrow x_1 = At + B$ } modo traslacional

Restando (1)-(2):

$$\ddot{\phi}_1 - \ddot{\theta}_2 = -2\omega_0^2(2\phi_1 - 2\theta_2)$$

llamando $x_2 = \phi_1 - \theta_2$:

$$\ddot{x}_2 = -\underbrace{4\omega_0^2}_{\omega^2} x_2$$

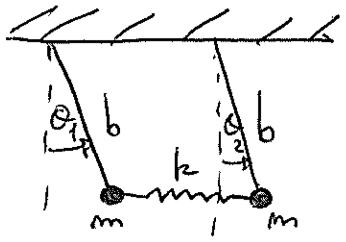
$\Rightarrow x_2 = C\cos(\omega t) + D\,\mathrm{sen}(\omega t)$ con $\omega = 2\omega_0^2$ } modo anti-simétr.

Como $x_1 = \phi_1 + \theta_2$: $\vec{e}_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$x_2 = \phi_1 - \theta_2$: $\vec{e}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}\phi_1\\\theta_2\end{pmatrix} = (\underset{\downarrow\ \frac{A}{2}}{A^*}t + \underset{\downarrow\ \frac{B}{2}}{B^*})\begin{pmatrix}1\\1\end{pmatrix} + (\underset{\downarrow\ \frac{C}{2}}{C^*}\cos(2\omega_0 t) + \underset{\downarrow\ \frac{D}{2}}{D^*}\mathrm{sen}(2\omega_0 t))\begin{pmatrix}1\\-1\end{pmatrix}$$

P4



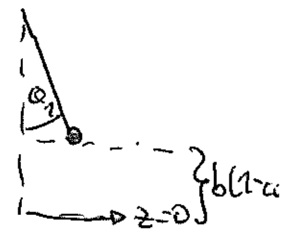
Encontrar modos normales

Sol:

Con energía: $K = \frac{m}{2} b^2 \dot{\theta}_1^2 + \frac{m}{2} b^2 \dot{\theta}_2^2$

$U_g = mgb(1-\cos\theta_1) + mgb(1-\cos\theta_2)$

$U_k = \frac{k}{2}(b\,\text{sen}\,\theta_2 - b\,\text{sen}\,\theta_1)^2$



Para pequeñas oscilaciones: $\text{sen}\,\theta \approx \theta$
$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$\Rightarrow E = \frac{m}{2} b^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + \frac{mgb}{2}(\theta_1^2 + \theta_2^2) + \frac{kb^2}{2}(\theta_2 - \theta_1)^2 \quad / \frac{d}{dt}()$$

$$0 = mb^2(\dot{\theta}_1 \ddot{\theta}_1 + \dot{\theta}_2 \ddot{\theta}_2) + mgb(\theta_1\dot{\theta}_1 + \theta_2\dot{\theta}_2) + kb^2(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)$$

Las dos ecuaciones:

$\frac{\partial}{\partial \dot{\theta}_1}$: $mb^2\ddot{\theta}_1 + mgb\theta_1 - kb^2(\theta_2 - \theta_1) = 0$

$$\ddot{\theta}_1 = -\left[\left(\frac{g}{b} + \frac{k}{m}\right)\theta_1 - \frac{k}{m}\theta_2\right]$$

$$= -\left[(\omega_p^2 + \omega_r^2)\theta_1 - \omega_r^2\theta_2\right] \quad \text{con } \omega_p^2 = \frac{g}{b},\ \omega_r^2 = \frac{k}{m}$$

$\frac{\partial}{\partial \dot{\theta}_2}$: $mb^2\ddot{\theta}_2 + mgb\theta_2 + kb^2(\theta_2 - \theta_1) = 0$

$$\ddot{\theta}_2 = -\left[\left(\frac{g}{b} + \frac{k}{m}\right)\theta_2 - \frac{k}{m}\theta_1\right]$$

$$= -\left[-\omega_r^2\theta_1 + (\omega_p^2 + \omega_r^2)\theta_2\right]$$

en forma matricial:

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = -\underbrace{\begin{pmatrix}\omega_p^2+\omega_r^2 & -\omega_r^2\\ -\omega_r^2 & \omega_p^2+\omega_r^2\end{pmatrix}}_{M}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

Sacamos los valores propios:

$$\begin{vmatrix}\omega_p^2+\omega_r^2-\omega^2 & -\omega_r^2\\ -\omega_r^2 & \omega_p^2+\omega_r^2-\omega^2\end{vmatrix} = (\omega_p^2+\omega_r^2-\omega^2)^2-\omega_r^4=0$$

$$\omega_p^4+\cancel{\omega_r^4}+\omega^4+2\omega_p^2\omega_r^2-2\omega_p^2\omega^2-2\omega_r^2\omega^2-\cancel{\omega_r^4}=0$$

$$\omega^4-2(\omega_p^2+\omega_r^2)\omega^2+\omega_p^4+2\omega_p^2\omega_r^2=0$$

$$\Rightarrow \quad \omega^2=\frac{\cancel{2}(\omega_p^2+\omega_r^2)\pm\sqrt{\cancel{4}(\omega_p^2+\omega_r^2)^2-\cancel{4}\omega_p^4-\cancel{8}^{2}\omega_p^2\omega_r^2}}{\cancel{2}}=$$

$$=\omega_p^2+\omega_r^2\pm\sqrt{\cancel{\omega_p^4}+\cancel{2\omega_p^2\omega_r^2}+\omega_r^4-\cancel{\omega_p^4}-\cancel{2\omega_p^2\omega_r^2}}=$$

$$=\omega_p^2+\omega_r^2\pm\omega_r^2$$

entonces los valores propios:

$$\boxed{\omega_+^2=\omega_p^2+2\omega_r^2} \qquad \boxed{\omega_-^2=\omega_p^2}$$

Ahora los vectores propios:

ω_+: $$\begin{pmatrix}\omega_p^2+\omega_r^2 & -\omega_r^2\\ -\omega_r^2 & \omega_p^2+\omega_r^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=(\omega_p^2+2\omega_r^2)\begin{pmatrix}x\\ y\end{pmatrix}$$

1^{ra} fila $\rightarrow$ $(\cancel{\omega_p^2}+\omega_r^2)x-\omega_r^2y=(\cancel{\omega_p^2}+2\omega_r^2)x \rightarrow -\omega_r^2y=\omega_r^2x \Rightarrow \boxed{\vec{e}_+=\begin{pmatrix}1\\ -1\end{pmatrix}}$

$\underline{\omega_-}$: $\begin{pmatrix} \omega_p^2+\omega_r^2 & -\omega_r^2 \\ -\omega_r^2 & \omega_p^2+\omega_r^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \omega_p^2 \begin{pmatrix} x \\ y \end{pmatrix}$

$1^{\underline{a}}$ fila $\longrightarrow$ $(\cancel{\omega_p^2}+\omega_r^2)x - \omega_r^2 y = \cancel{\omega_p^2} x$

$\omega_r^2 x = \omega_r^2 y \Rightarrow x = y \Rightarrow \boxed{\vec{e}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$